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REVERSE MATHEMATICS AND Π^1_2 COMPREHENSION

CARL MUMMERT* AND STEPHEN G. SIMPSON†

Abstract. We initiate the reverse mathematics of general topology. We show that a certain metrization theorem is equivalent to Π^1_2 comprehension. An *MF space* is defined to be a topological space of the form $\mathrm{MF}(P)$ with the topology generated by $\{N_p \mid p \in P\}$. Here P is a poset, $\mathrm{MF}(P)$ is the set of maximal filters on P, and $N_p = \{F \in \mathrm{MF}(P) \mid p \in F\}$. If the poset P is countable, the space $\mathrm{MF}(P)$ is said to be *countably based*. The class of countably based MF spaces can be defined and discussed within the subsystem ACA_0 of second order arithmetic. One can prove within ACA_0 that every complete separable metric space is homeomorphic to a countably based MF space which is regular. We show that the converse statement, "every countably based MF space which is regular is homeomorphic to a complete separable metric space," is equivalent to Π^1_2-CA_0. The equivalence is proved in the weaker system Π^1_1-CA_0. This is the first example of a theorem of core mathematics which is provable in second order arithmetic and implies Π^1_2 comprehension.

In the foundations of mathematics, there is an ongoing research program known as *reverse mathematics*. One focuses on specific core mathematical theorems τ, and one determines the weakest set existence axioms which are needed in order to prove τ. Such determinations are made in the context of subsystems of second order arithmetic, Z_2. The strength of τ is measured by showing that τ is logically equivalent to a particular subsystem of Z_2, over a weaker subsystem. The standard reference for reverse mathematics and subsystems of Z_2 is Simpson [12]. See also [11].

Previous reverse mathematics studies [12], [11] have included an extensive development of the reverse mathematics of complete separable metric spaces. We now initiate the reverse mathematics of general topological spaces.

Definition 1. The subsystems of Z_2 used in this paper are ACA_0, Π^1_1-CA_0, and Π^1_2-CA_0. We briefly describe these systems. The language of Z_2 includes *number variables* $k, m, n, \ldots$ ranging over $\mathbb{N}$, the set of natural numbers, and *set variables* $X, Y, Z, \ldots$ ranging over subsets of $\mathbb{N}$. All of our systems include

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first order arithmetic plus the induction axiom

$$\forall X\,[(0 \in X \wedge \forall n\,(n \in X \Rightarrow n+1 \in X)) \Rightarrow \forall n\,(n \in X)].$$

The system ACA_0 includes *comprehension axioms*

$$\exists X\,\forall n\,[n \in X \Leftrightarrow \varphi(n)]$$

where the formula $\varphi(n)$ does not mention X and is *arithmetical*, i.e., it contains no set quantifiers. The system Π^1_1-CA_0 includes comprehension for formulas $\varphi(n)$ which are Π^1_1, i.e., of the form $\forall Y\,\theta(n, Y)$ where $\theta(n, Y)$ is arithmetical. The system Π^1_2-CA_0 includes comprehension for formulas $\varphi(n)$ which are Π^1_2, i.e., of the form $\forall Y\,\exists Z\,\theta(n, Y, Z)$ where $\theta(n, Y, Z)$ is arithmetical. In each of these systems, $\varphi(n)$ is allowed to mention *parameters*, i.e., free set variables. Details concerning these systems are in [12, Sections I.1–I.6].

REMARK 1. In the following definitions, we shall show how to formalize some key concepts of general topology within ACA_0. As is usual in reverse mathematics (see for example the discussion of complete separable metric spaces in [12, Sections II.5–II.6]), we shall employ "coding" via definitional extensions of the language of Z_2. In each case, the "code" is a natural number or a set of natural numbers, but the encoded object may be uncountable. It will be always be clear how to convert our informal "coding" definitions into precise, rigorous, definitional extensions of ACA_0.

DEFINITION 2. A *poset* is a partially ordered set. Within ACA_0, let P be a countable poset. A *filter*[1] on P is a set $F \subseteq P$ such that (1) for all $p \in F$ and $q \geq p$ we have $q \in F$, and (2) for all $p, q \in F$ there exists $r \in F$ such that $p \geq r$ and $q \geq r$. A *maximal filter* is a filter which is not included in any larger filter. Within ACA_0, consider the topological space $\mathrm{MF}(P)$ whose points are the maximal filters on P, and whose basic open sets are

$$N_p = \{F \in \mathrm{MF}(P) \mid p \in F\}$$

for all $p \in P$. Such a space is called a *countably based MF space*. See also Mummert [8], [9].

REMARK 2. In Definition 2 within ACA_0, the countable poset P is regarded as a code for the topological space $\mathrm{MF}(P)$, and each $p \in P$ is regarded as a code for the basic open neighborhood N_p in $\mathrm{MF}(P)$. In general, the open sets of $\mathrm{MF}(P)$ are of the form

$$U_L = \bigcup_{p \in L} N_p = \{F \in \mathrm{MF}(P) \mid \exists p \in L\,[F \in N_p]\}$$

where L is a subset of P. Thus, if F is a *point* of $\mathrm{MF}(P)$, i.e., a maximal filter on P, then we have $F \in N_p$ if and only if $p \in F$, and $F \in U_L$ if and

[1] Our notion of a filter on a poset is identical to the one that is often used in forcing in axiomatic set theory. See for example Kunen [7].

only if $F \cap L \neq \emptyset$. Thus the countable set $L \subseteq P$ may be regarded as a code for the open set $U_L \subseteq \mathrm{MF}(P)$.

Definition 3. Within ACA_0, if (A, d) is a countable pseudometric space, let $(\widehat{A}, \widehat{d})$ be the complete separable metric space whose points are the equivalence classes of Cauchy sequences in (A, d), two such sequences $\langle a_n \mid n \in \mathbb{N}\rangle$ and $\langle b_n \mid n \in \mathbb{N}\rangle$ being equivalent if and only if $\lim_n d(a_n, b_n) = 0$. Clearly all complete separable metric spaces arise in this way. See also [12, Sections I.4, II.5–II.7, III.2, IV.1–IV.2]. Given (A, d) as above, consider also the countable poset

$$P_{(A,d)} = A \times \mathbb{Q}^+$$

where $\mathbb{Q}^+$ is the set of positive rational numbers, partially ordered by putting $(a, r) < (b, s)$ if and only if $d(a, b) + r < s$. One can show in ACA_0 (see [8, Sections 2.3.1 and 3.2] and [9]) that there is a canonically arithmetically definable, one-to-one correspondence between the points of $(\widehat{A}, \widehat{d})$ and the maximal filters on $P_{(A,d)}$. Moreover, a point $x \in \widehat{A}$ belongs to a basic open ball

$$B(a, r) = \left\{ x \in \widehat{A} \mid \widehat{d}(a, x) < r \right\}$$

if and only if the corresponding maximal filter belongs to $N_{(a,r)}$. Thus, in the topological context, we are justified in identifying the complete separable metric space $(\widehat{A}, \widehat{d})$ with the countably based MF space $\mathrm{MF}(P)$, where $P = P_{(A,d)}$.

Definition 4. Within ACA_0, if P and Q are countable posets, we define a *code for a continuous function from MF*(P) *to MF*(Q) to be a set $\Phi \subseteq P \times Q$ such that, for all maximal filters F on P,

$$\Phi(F) = \{q \in Q \mid \exists p \in F\, [(p, q) \in \Phi]\}$$

is a maximal filter on Q. It can be shown that each such code induces a continuous function $\Phi\colon \mathrm{MF}(P) \to \mathrm{MF}(Q)$. Moreover, all continuous functions from $\mathrm{MF}(P)$ to $\mathrm{MF}(Q)$ are induced by such codes. A *homeomorphism from MF*(P) *to MF*(Q) is a coded continuous $\Phi\colon \mathrm{MF}(P) \to \mathrm{MF}(Q)$ together with a coded continuous inverse $\Phi^{-1}\colon \mathrm{MF}(Q) \to \mathrm{MF}(P)$. The requirement of a coded continuous inverse is apparently not superfluous; we do not know whether it is provable in Π^1_2-CA_0 that every coded continuous open bijection $\Phi\colon \mathrm{MF}(P) \to \mathrm{MF}(Q)$ has a coded continuous inverse. See also [8, Section 3.2.3] and [9].

Remark 3. We offer the following reasons for focusing on the class of countably based MF spaces. See also Mummert [8], [9].

1. In earlier reverse mathematics studies (see [12], [11]), the restriction to subsystems of Z_2 has been appropriate, natural, and fruitful. Therefore,

in order to extend reverse mathematics to general topology, it is important to identify a class of topological spaces which is reasonably broad yet conveniently formalizable in Z_2. It turns out that the countably based MF spaces are just such a class.

2. As already noted in Definition 3, the class of countably based MF spaces includes all complete separable metric spaces. Thus, it includes many of the topological spaces which arise in core mathematical disciplines such as analysis and geometry.
3. In addition, the class of countably based MF spaces includes many topological spaces which are not metrizable. An interesting example is the Gandy/Harrington space (see [3, p. 240]), which is well known to be of great importance in contemporary descriptive set theory [1], [4]. See also [8, Section 2.3.4] and [9].
4. The class of countably based MF spaces enjoys some nice closure properties. First, the product of countably many countably based MF spaces is homeomorphic to a countably based MF space. Second, any nonempty G_δ set[2] in a countably based MF space is homeomorphic to a countably based MF space. See also [8, Section 2.3.2] and [9].

Definition 5. A topological space is said to be *regular* if, for every open set U and point $x \in U$, there exists an open set V such that $x \in V$ and the closure of V is included in U. See for example Kelley [5, page 113]. It is well known and easy to see that metric spaces are regular.

Definition 6. We study the reverse mathematics of the following metrization theorem for countably based MF spaces.

MFMT: A countably based MF space is homeomorphic to a complete separable metric space if and only if it is regular.

Lemma 1. MFMT is provable in Π^1_2-CA_0.

Proof. One direction of MFMT is easy. It is straightforward to prove in ACA_0 that every countably based MF space which is homeomorphic to a complete separable metric space is regular.

In the other direction, we prove within Π^1_2-CA_0 that every countably based MF space which is regular is homeomorphic to a complete separable metric space. Our argument is loosely based on the original proofs of the metrization theorems due to Urysohn (see also Schröder [10]) and Choquet (see also Kechris [3, Section 8.E]). The details of our argument are in Mummert [8, Section 4.3] and [9]. Here we provide only a sketch.

[2] In the context of arbitrary topological spaces, a G_δ *set* is defined to be the intersection of countably many open sets. In metric spaces, it is easy to see that every closed set is a G_δ set, but unfortunately this result does not generalize to arbitrary topological spaces, or even to arbitrary countably based MF spaces. As an example in the Gandy/Harrington space, we may take any lightface Π^1_1 set which is not boldface Σ^1_1. See also [8, proof of Theorem 2.3.40] and [9].

Reasoning within Π^1_2-CA_0, let P be a countable poset such that $\mathrm{MF}(P)$ is regular. We use Π^1_2 comprehension to form the set of ordered pairs $(p, q) \in P \times P$ such that N_p includes the closure of N_q. Using this set as a parameter, we adapt a construction of Schröder in effective topology [10] to obtain a metric d_1 on $\mathrm{MF}(P)$ which is compatible with the topology of $\mathrm{MF}(P)$. Thus $\mathrm{MF}(P)$ is metrizable, but we have not yet shown that $\mathrm{MF}(P)$ is completely metrizable.

Fix a countable dense set $A \subseteq \mathrm{MF}(P)$. We use Π^1_2 comprehension to form the sets

$$\{(a, r, p) \in A \times \mathbb{Q}^+ \times P \mid B_1(a, r) \subseteq N_p\}$$

and

$$\{(a, r, p) \in A \times \mathbb{Q}^+ \times P \mid N_p \subseteq B_1(a, r)\}$$

where $B_1(a, r) = \{x \in \mathrm{MF}(P) \mid d_1(a, x) < r\}$. Using these sets as parameters, we attempt to imitate the proof of Choquet's theorem (see [3, Section 8.E]) that any metric space having the strong Choquet property is completely metrizable. It is provable in ZFC (see [8, Theorem 2.3.29] and [9]) that every MF space has the strong Choquet property, but unfortunately this game-theoretic property is not definable in Z_2. We overcome this obstacle by giving a direct proof within Π^1_2-CA_0 that every countably based MF space which is metrizable is completely metrizable. The details are in [8, Section 4.3] and [9].

In this way we obtain a complete metric d_2 on $\mathrm{MF}(P)$ which is compatible with d_1 and hence compatible with the topology of $\mathrm{MF}(P)$. It is then straightforward using Π^1_2 comprehension to construct a homeomorphism between $\mathrm{MF}(P)$ and $(\widehat{A}, \widehat{d_2}) = \mathrm{MF}(Q)$ where $Q = P_{(A,d_2)}$. This proves our lemma. $\dashv$

Theorem 1. The following are equivalent over Π^1_1-CA_0.

1. Π^1_2 comprehension.
2. MFMT.

Proof. Lemma 1 shows that Π^1_2 comprehension implies MFMT. It remains to prove the reversal. We work in Π^1_1-CA_0 and assume MFMT. Consider a Σ^1_2 formula $\exists X\, \psi(n, X)$ where $\psi(n, X)$ is Π^1_1. In order to prove Π^1_2 comprehension, it suffices to prove the existence of the set $S = \{n \in \mathbb{N} \mid \exists X\, \psi(n, X)\}$. By Kondo's Π^1_1 uniformization theorem (provable in Π^1_1-CA_0, see [12, Section VI.2]), we may safely assume that for each n there exists at most one X, call it X_n, such that $\psi(n, X)$ holds.

We write $\psi(n, X)$ in normal form as $\neg \exists f\, \forall m\, R(n, X[m], f[m])$. Here we are using the finite sequence notation $X[m] = \langle X(0), \ldots, X(m-1)\rangle$ and $f[m] = \langle f(0), \ldots, f(m-1)\rangle$. We may safely assume that $\forall n\, R(n, \langle\rangle, \langle\rangle)$ holds. For finite sequences σ and τ, we write $\sigma \supset \tau$ if and only if τ is a proper initial segment of σ. Let P be the countable poset consisting of all ordered triples $(n, X[k], f[k])$ such that $(\forall m \leq k)\, R(n, X[m], f[m])$ holds, plus all ordered pairs $(n, X[k])$, partially ordered as follows:

1. $(n, X[k], f[k]) < (n', X'[k'], f'[k'])$ if and only if $n = n'$ and $X[k] \supset X'[k']$ and $f[k] \supset f'[k']$.
2. $(n, X[k]) < (n', X'[k'])$ if and only if $n = n'$ and $X[k] \supset X'[k']$.
3. $(n, X[k], f[k]) < (n', X'[k'])$ if and only if $n = n'$ and $X[k] \supset X'[k']$.
4. $(n, X[k]) < (n', X'[k'], f'[k'])$ never.

The maximal filters F on P are of three types:

1. $F = \{p \in P \mid q \leq p\}$ where q is a minimal element of P.
2. $F = \{(n, X[k], f[k]), (n, X[k]) \mid k \in \mathbb{N}\}$ where n, X, f are such that $\forall m\, R(n, X[m], f[m])$ holds.
3. $F = F_n = \{(n, X_n[k]) \mid k \in \mathbb{N}\}$ where n is such that $\exists X\, \psi(n, X)$.

Note that there is a closed set $C \subseteq \mathrm{MF}(P)$ consisting of F_n for all n such that $\exists X\, \psi(n, X)$ holds. The complement of C is the open set $\bigcup_{n \in \mathbb{N}} N_{(n, \langle\rangle, \langle\rangle)}$.

We claim that $\mathrm{MF}(P)$ is regular. Clearly it will suffice to show that, for all maximal filters F and basic open sets N_p such that $F \in N_p$, we can find a basic open set N_q such that $F \in N_q$ and the closure of N_q is included in N_p. We shall find such an N_q with the additional property that N_q is closed. Thus we shall see that the topology of $\mathrm{MF}(P)$ is generated by basic open sets which are also closed.

Case 1: $F = \{p \in P \mid q \leq p\}$ where q is a minimal element of P. In this case F is an isolated point of $\mathrm{MF}(P)$, and $N_q = \{F\}$ is closed. For use in cases 2 and 3, let W be the open set consisting of these isolated points, i.e.,

$$W = \bigcup_{q \in M} N_q$$

where M is the set of minimal elements of P.

Case 2: $F = \{(n, X[k], f[k]), (n, X[k]) \mid k \in \mathbb{N}\}$ where n, X, f are such that $\forall m\, R(n, X[m], f[m])$ holds. It follows that $\psi(n, X)$ fails. Hence $X \neq X_n$ if X_n exists. Suppose $F \in N_p$, i.e., $p \in F$. If $p = (n, X[k])$, then N_p is closed, because the complement of N_p is the open set

$$(W \setminus N_p) \cup \bigcup_{p' \in L'} N_{p'}$$

where L' is the set of $p' = (n', X'[k]) \in P$ such that $n' \neq n$ or $X'[k] \neq X[k]$. If $p = (n, X[k], f[k])$, let $m \geq k$ be so large that $X[m] \neq X_n[m]$ if X_n exists, and put $q = (n, X[m], f[m])$. Then $F \in N_q \subseteq N_p$. Moreover, N_q is closed, because the complement of N_q is the open set

$$(W \setminus N_q) \cup \bigcup_{p' \in L''} N_{p'} \cup \bigcup_{q' \in M''} N_{q'}$$

where L'' is the set of $p' = (n', X'[m])$ such that $n' \neq n$ or $X'[m] \neq X[m]$, and M'' is the set of $q' = (n', X'[m], f'[m]) \in P$ such that $n' \neq n$ or $X'[m] \neq X[m]$ or $f'[m] \neq f[m]$.

Case 3: $F = F_n$. In this case, for all N_p such that $F \in N_p$, we have $p = (n, X_n[k])$ for some k. As in case 2, N_p is closed.

We have now proved that $\mathrm{MF}(P)$ is regular. It follows by MFMT that there exists a homeomorphism Φ of $\mathrm{MF}(P)$ onto a complete separable metric space $(\widehat{A}, \widehat{d})$. In particular, since C is a closed set in $\mathrm{MF}(P)$, $\Phi(C)$ is a closed set in $(\widehat{A}, \widehat{d})$. For each $n \in \mathbb{N}$, if F_n exists then F_n is the unique point of $C \cap N_{(n,\langle\rangle)}$, hence $\Phi(F_n) \in \widehat{A}$ is the unique point of $\Phi(C) \cap \Phi(N_{(n,\langle\rangle)})$. If F_n does not exist, then $C \cap N_{(n,\langle\rangle)} = \emptyset$, hence $\Phi(C) \cap \Phi(N_{(n,\langle\rangle)}) = \emptyset$. Note also that the sequence of open sets $\Phi(N_{(n,\langle\rangle)}) \subseteq \widehat{A}$, $n \in \mathbb{N}$, is arithmetically definable uniformly in n, using the code of Φ^{-1} as a parameter. Thus we can use Π^1_1 comprehension to form the set

$$S = \{n \mid \Phi(C) \cap \Phi(N_{(n,\langle\rangle)}) \neq \emptyset\} = \{n \mid F_n \text{ exists}\} = \{n \mid \exists X\ \psi(n, X)\}.$$

This completes the proof. ⊣

REMARK 4. Theorem 1 shows that a certain metrization theorem is logically equivalent to Π^1_2-CA_0 over Π^1_1-CA_0. We believe that this is the first convincing instance of a core mathematical theorem which is equivalent to Π^1_2 comprehension, in the sense of reverse mathematics. Previous reverse mathematics results within Z_2 (see [12], [11]) have involved only set existence axioms which are strictly weaker than Π^1_2 comprehension.[3]

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DEPARTMENT OF MATHEMATICS
PENNSYLVANIA STATE UNIVERSITY
E-mail: mummert@math.psu.edu
URL: http://www.math.psu.edu/mummert/

DEPARTMENT OF MATHEMATICS
PENNSYLVANIA STATE UNIVERSITY
E-mail: simpson@math.psu.edu
URL: http://www.math.psu.edu/simpson/