Reverse mathematics of MF spaces

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Abstract

This paper gives a formalization of general topology in second order arithmetic using countably based MF spaces. This formalization is used to study the reverse mathematics of general topology.

For each poset P we let $\mathrm{MF}(P)$ denote the set of maximal filters on P endowed with the topology generated by $\{N_p \mid p \in P\}$, where $N_p = \{F \in \mathrm{MF}(P) \mid p \in F\}$. Define a countably based MF space to be a space of the form $\mathrm{MF}(P)$ for some countable poset P. The class of countably based MF spaces includes all complete separable metric spaces as well as many nonmetrizable spaces.

The following reverse mathematics results are obtained. The proposition that every nonempty G_δ subset of a countably based MF space is homeomorphic to a countably based MF space is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ over ACA_0. The proposition that every uncountable closed subset of a countably based MF space contains a perfect set is equivalent over $\Pi^1_1\text{-}\mathsf{CA}_0$ to the proposition that $\aleph_1^{L(A)}$ is countable for all $A \subseteq \mathbb{N}$. The proposition that every regular countably based MF space is homeomorphic to a complete separable metric space is equivalent to $\Pi^1_2\text{-}\mathsf{CA}_0$ over $\Pi^1_1\text{-}\mathsf{CA}_0$.

1 Introduction

In this paper, countably based MF spaces are used to formalize general topology in second-order arithmetic. This formalization is used to obtain reverse mathematics results and set-theoretic independence results.

A *countably based MF space* is the set $\mathrm{MF}(P)$ of maximal filters on a countable poset P. The topology on $\mathrm{MF}(P)$ is generated by $\{N_p \mid p \in P\}$, where $N_p = \{F \in \mathrm{MF}(P) \mid p \in F\}$. Mummert and Stephan have shown [15] that the countably based MF spaces are precisely the second-countable T_1 spaces with the strong Choquet property (see Definition 2.7). These spaces form a rich class, including all complete separable metric spaces as well as nonmetrizable spaces such as the Gandy–Harrington space. They are thus a natural setting in which to study reverse mathematics.

1.1 Summary of results

Section 2 presents the formalization of countably based MF spaces and contains several reverse mathematics results. The section begins with basic definitions and important examples (complete separable metric spaces and the Gandy-Harrington space). In Section 2.2, we prove an analogue of the Baire Category Theorem for countably based MF spaces in ACA_0. We then show that Π^1_1-CA_0 is required to prove that every filter on a countable poset is included in a maximal filter. In Section 2.3, we show that every countably based MF space has the strong Choquet property and discuss the implications of this result. Section 2.4 considers the closure properties of the class of countably based MF spaces. We show that ACA_0 can form countable products of countably based poset spaces. Countably based MF spaces are closed under taking G_δ subsets, and this is equivalent to Π^1_1-CA_0 over ACA_0. Section 2.5 is devoted to a technical lemma showing that coanalytic subsets of $\mathbb{N}^\mathbb{N}$ may be represented as closed subsets of countably based MF spaces. This representation is crucial to several results later in the paper.

In Section 3, we show in ZFC that a countably based Hausdorff MF space is either countable or contains a perfect closed set. We obtain a reversal of a weaker statement to ATR_0 over ACA_0. We then show that the principle

> Every closed subset of a countably based Hausdorff MF space is either countable or contains a perfect closed set.

is equivalent to the proposition that $\aleph_1^{L(A)}$ is countable for each $A \subseteq \mathbb{N}$.

In Section 4, we study metrization theorems. Section 4.1 proves an analogue of Urysohn's metrization theorem for countably based MF spaces and gives a proof of the following statement in in Π^1_2-CA_0.

> MFMT: A countably based MF space is homeomorphic to a complete separable metric space if and only if it is regular.

In Section 4.2, we show that MFMT is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0. This is the first example of a natural mathematical statement which implies Π^1_2-CA_0. Previous results involving Π^1_2-CA_0 have been conservation results rather than full equivalences. Simpson (see Chapter VII of [18]) has established conservation results that apply to Π^1_2-CA_0. Rathjen has conducted an ordinal analysis of Π^1_2-CA_0, which is described in [16]. Heinatsch and Möllerfeld have shown [3] that a certain determinacy scheme proves the same Π^1_1 sentences as Π^1_2-CA_0. Möllerfeld has shown [12] that Π^1_2-CA_0 proves the same Π^1_1 sentences as a certain system of inductive definitions.

1.2 Reverse mathematics and second-order arithmetic

A typical reverse mathematics result consists of a proof of a theorem T in a subsystem S of second-order arithmetic and a proof (called a reversal) that T implies S if a weaker base system is assumed. The following subsystems of second-order arithmetic are used in this paper. RCA_0 contains Δ^0_1 comprehension and Σ^0_1 induction. ACA_0 contains arithmetic comprehension and arithmetic induction. ACA_0^+ is ACA_0 plus an axiom stating that any arithmetical functional may be iterated along the well-ordering $\mathbb{N}$. ATR_0 is ACA_0 plus an axiom stating that any arithmetical functional may be iterated along any countable well-ordering. Π^1_k-CA_0, for $k = 1$ or 2, is ACA_0 plus Π^1_k comprehension. The standard reference for subsystems of second-order arithmetic, which gives precise definitions of the subsystems just mentioned, is by Simpson [18].

The base system for many results in this paper is ACA_0, although the majority of previous reverse mathematics results use the base system RCA_0. We use ACA_0 as a base system because we formalize countably based MF spaces in ACA_0. There are two central difficulties in formalizing countably based MF spaces in RCA_0. The first difficulty is that RCA_0 is not able to form the upward closure of an arbitrary subset of a countable poset. It is possible to avoid this problem by coding a filter by an enumerated descending sequence whose upward closure is the filter; such a formalization of countably based MF spaces in RCA_0 is given in [13]. The second, more serious, difficulty is that RCA_0 does not prove that every countable poset has a maximal filter, or even that for every countable poset there is an enumeration of a maximal filter on the poset. These results follow from joint work with Steffen Lempp that will appear in [8]. The two difficulties just discussed indicate that RCA_0 is not strong enough to give an honest formalization of countably based MF spaces.

Our main justification for using countably based MF spaces, which seem to require ACA_0 for an honest formalization, is that they form a natural class of spaces and that they are amenable to formalization in second-order arithmetic. The reverse mathematics results in this paper show that specific theorems of general topology, formalized using countably based MF spaces, are equivalent to strong systems such as Π^1_1-CA_0 and Π^1_2-CA_0 or are independent of ZFC. ACA_0 is weak enough to serve as a base system for reversals such as these.

2 Formalization of MF spaces

2.1 Definitions and examples

In this section, we give definitions which formalize countably based MF spaces in ACA_0. Our definition of posets and filters is the same as in axiomatic set theory as described by Kunen [6].

Definition 2.1. The following definitions are made in RCA_0.[1] A *countable poset* is a pair $\langle P, \preceq \rangle$, where $P \subseteq \mathbb{N}$ and $\preceq$ is a (coded) binary relation on P satisfying the following requirements.

1. $p \preceq p$ for all $p \in P$.
2. If $p \preceq q$ and $q \preceq p$ then $q = p$.
3. If $p \preceq q$ and $q \preceq r$ then $p \preceq r$.

If $p \preceq q$ we say p *extends* q; we write $p \perp q$ if p and q have no common extension.

A *filter* is a subset F of a countable poset P satisfying the following conditions:

1. If $p \in F$ and $p \preceq q$ then $q \in F$.
2. If $p \in F$ and $q \in F$ then there is an $r \in F$ such that $r \preceq p$ and $r \preceq q$.

If F, G are filters on the a poset P and $F \subseteq G$ then we say G *extends* F. A filter is *maximal* if it cannot be extended to a strictly larger filter.

[1] Throughout this paper, a phrase such as this indicates that the definition may be formalized as a definitional extension of the specified subsystem of second-order arithmetic.

Definition 2.2. The following definition is made in ACA_0. Let P be a countable poset. We let $\mathrm{MF}(P)$ denote the collection of all maximal filters on P. For each $p \in P$, we let

$$N_p = \{x \in \mathrm{MF}(P) \mid p \in x\}.$$

For each $U \subseteq P$, we let

$$N_U = \{x \in \mathrm{MF}(P) \mid \exists p \in U(x \in N_p)\}.$$

We will call a set of the form N_U an *open set* of $\mathrm{MF}(P)$, and regard U as a code for N_U. This topological language is justified because ACA_0 proves that the open sets of $\mathrm{MF}(P)$ are closed under unions and finite intersections. The topology containing these open sets will be called the *poset topology* on $\mathrm{MF}(P)$.

A code for a closed set $C \subseteq \mathrm{MF}(P)$ is the same as a code for the open set $\mathrm{MF}(P) \setminus C$.

Definition 2.3. The following definition is made in ACA_0. A topological space of the form $\mathrm{MF}(P)$, with the topology defined in Definition 2.2, is called a *countably based MF space*.

Remark 2.1. We define $\mathrm{MF}(P)$ to consist of all the maximal filters on P rather than just the generic maximal filters. This is because the nongeneric points may form an important part of the topology. Indeed, for any poset P the subspace of $\mathrm{MF}(P)$ consisting of generic points is totally disconnected, but we are interested in spaces which may be connected.

Remark 2.2. It is obvious that every countably based MF is a second countable topological space. Frank Stephan has proved the following closely related result in ZFC. For each poset P such that $\mathrm{MF}(P)$ is a second countable space there is a countable subposet R of P such that $\mathrm{MF}(P)$ is homeomorphic to $\mathrm{MF}(R)$ by the map $f \mapsto f \cap R$. This result, which will appear in [15], gives added justification to the term "countably based poset space."

For any subset F of a poset P, we let $\mathrm{ucl}(F)$ denote the *upward closure* of F, that is, the set $\{p \in P \mid \exists q \in F(q \preceq p)\}$. ACA_0 proves that the upward closure of any set exists.

Recall that if A is a countable set and d is a pseudometric on A then the collection $\widehat{A}$ of Cauchy sequences on A forms a complete separable metric space with a unique complete metric $\hat{d}$ such that $\hat{d}$ equals d on A, and every complete separable metric space arises in this way (see Section II.5

of [18]). The next theorem shows that every complete separable metric space is canonically representable as a countably based MF space. Similar constructions have been used by Lawson [7], to represent each complete separable metric space as the space of total objects of a domain with the Scott topology, and by Simpson (see Definition II.6.1 of [18]), to represent continuous functions between complete separable metric spaces.

Theorem 2.1. *The following is provable in* ACA_0. *Let* $\langle \widehat{A}, \hat{d} \rangle$ *be a complete separable metric space. There is a canonical countable poset* P *and a canonical arithmetically definable correspondence between* $\widehat{A}$ *and* $\mathrm{MF}(P)$ *such that the metric topology on* $\widehat{A}$ *corresponds to the topology on* $\mathrm{MF}(P)$.

Proof. Let P be the set of pairs $\langle a, r \rangle$ with $a \in A$ and $r \in \mathbb{Q}^+$. Order P by putting $\langle a', r' \rangle \preceq \langle a, r \rangle$ if and only if $d(a, a') + r' < r$. This poset may be formed in ACA_0, because the order relation on P is definable by a Σ^0_1 formula relative to d.

We define a functional Φ from Cauchy sequences on A to subsets of P. Let $\langle a_i \rangle$ be a Cauchy sequence on A; by taking a canonical subsequence, we may assume that $d(a_i, a_j) < 2^{-i}$ for all $i \leq j$. Let $\Phi(\langle a_i \rangle) = \mathrm{ucl}\{\langle a_i, 2^{-i} \rangle\} \subseteq P$. It is clear that $\Phi(\langle a_i \rangle)$ is a filter on P if $\langle a_i \rangle$ is a Cauchy sequence.

It can be shown that if $\langle a_i \rangle$ and $\langle b_i \rangle$ are Cauchy sequences such that $\lim_i d(a_i, b_i) = 0$ then $\Phi(\langle a_i \rangle) = \Phi(\langle b_i \rangle)$. This allows us to prove that $\Phi(\langle a_i \rangle)$ is a maximal filter for each Cauchy sequence $\langle a_i \rangle$.

It is straightforward to verify that Φ is a continuous open bijection from $\widehat{A}$ to $\mathrm{MF}(P)$ and that Φ and its inverse are canonically arithmetically definable. □

The *Gandy–Harrington space* is the set $\mathbb{N}^\mathbb{N}$ with the topology generated by the lightface Σ^1_1 sets. This space is of great importance in modern descriptive set theory, as documented in [5], [10], and pp. 907–908 of [2]. The Gandy–Harrington space is interesting to us because it is a separable Hausdorff nonregular space, and is thus nonmetrizable.

Theorem 2.2. *The following is provable in* Π^1_1-CA_0. *There is a countable poset* P, *definable by a* Σ^1_1 *formula, such that there is a canonically* Σ^1_1-*definable correspondence between points of* $\mathrm{MF}(P)$ *and points of* $\mathbb{N}^\mathbb{N}$ *such that the topology on* $\mathrm{MF}(P)$ *corresponds to the Gandy–Harrington topology on* $\mathbb{N}^\mathbb{N}$.

Proof. The poset P consists of sequences $\langle \sigma, \langle T_1, \tau_1 \rangle, \ldots, \langle T_n, \tau_n \rangle \rangle$ where $\sigma \in \mathbb{N}^{<\mathbb{N}}$, each T_i is a computable tree on $\mathbb{N} \times \mathbb{N}$, $|\sigma| = |\tau_1| = \cdots = |\tau_n|$, and there is an $f \in \mathbb{N}^\mathbb{N}$ such that $\sigma \subseteq f$ and for each $i \leq n$ there is a $g_i \in \mathbb{N}^{\mathbb{N}}$ such that $\tau_i \subseteq g_i$ and $\langle f, g\rangle \in [T_i]$. We order P by setting $\langle \sigma, \langle T_1, \tau_1\rangle, \ldots, \langle T_n, \tau_n\rangle\rangle \prec \langle \sigma', \langle T_1', \tau_1'\rangle, \ldots, \langle T_m', \tau_m'\rangle\rangle$ if $\sigma' \subseteq \sigma$, $|\sigma| > |\sigma'|$, and for each $i \leq m$ there is a $j \leq n$ such that $T_j = T_i'$ and $\tau_i' \subseteq \tau_j$.

The proof of Theorem 2.3.38 in [13] shows that there is a Σ^1_1-definable homeomorphic correspondence between $\mathrm{MF}(P)$ and the Gandy–Harrington space. $\square$

Definition 2.4. The following definition is made in RCA_0. Let P and Q be countable posets. We define a *continuous function code* to be a subset of $P \times Q$. Each continuous function code F induces a partial function f from $\mathrm{MF}(P)$ to $\mathrm{MF}(Q)$ given by

$$f(x) = \mathrm{ucl}\{q \in Q \mid p \in x \text{ and } \langle p, q\rangle \in F\}.$$

We say that f is *defined* at $x \in \mathrm{MF}(P)$ if $f(x)$ is a maximal filter on Q. A *coded continuous function* is a total function induced by a continuous function code.

ACA_0 proves that the preimage of an open set under a coded continuous function is always open. Moreover, ACA_0 proves that for any coded continuous function f from $\mathrm{MF}(P)$ to $\mathrm{MF}(Q)$ there is a canonical arithmetical functional Φ sending coded open subsets of $\mathrm{MF}(Q)$ to coded open subsets of $\mathrm{MF}(P)$ such that $\Phi(V) = f^{-1}(V)$ for every coded open subset V of $\mathrm{MF}(Q)$.

Before proving that every continuous function is induced by a a continuous function code, we present two simple examples. Let P and Q be countable posets. The identity function $i\colon \mathrm{MF}(P) \to \mathrm{MF}(P)$ is induced by the code $\{\langle p, p\rangle \mid p \in P\}$. For each $y \in \mathrm{MF}(Q)$ the constant function $c_y\colon \mathrm{MF}(P) \to \mathrm{MF}(Q)$ sending every $x \in \mathrm{MF}(P)$ to y is induced by the code $\{\langle p, q\rangle \mid p \in P, q \in y\}$.

Theorem 2.3. *The following is provable in* ZFC. *Let P and Q be posets. Every continuous function from $\mathrm{MF}(P)$ to $\mathrm{MF}(Q)$ is induced by a continuous function code.*

Proof. Let $f\colon \mathrm{MF}(P) \to \mathrm{MF}(Q)$ be given. Define $F = \{\langle p, q\rangle \mid f(N_p) \subseteq N_q\}$. We will show that f is induced by F. Choose x, y with $f(x) = y$. We must show that $y = F[x]$, where $F[x] = \mathrm{ucl}\{q \mid \exists p \in x[\langle p, q\rangle \in F]\}$. Because f is continuous, for each $q \in y$ there is a $p \in x$ such that $f(N_p) \subseteq N_q$; thus $q \in F[x]$. Now suppose $q' \in F[x]$. By the definition of $F[x]$, there is a $p \in x$ and a $q'' \preceq q'$ such that $f(N_p) \subseteq N_{q''} \subseteq N_{q'}$. Now $x \in N_p$ and $f(x) = y$, so $y \in N_{q'}$; thus $q' \in y$. We have shown $y = F[x]$. $\square$

Definition 2.5. The following definition is made in ACA_0. We say that two countably based poset spaces X and Y are *homeomorphic* if there is a continuous coded bijection $f\colon X \to Y$ with a continuous coded inverse $f^{-1}\colon Y \to X$.

We say that a countably based MF space X is *homeomorphic to a complete separable metric space* if there is a complete separable metric space $\widehat{A}$ such that X is homeomorphic to the canonical poset representation of $\widehat{A}$ (as described in Theorem 2.1).

Remark 2.3. Alternative definitions of a homeomorphism between poset spaces may be considered. In particular, it is possible to define a homeomorphism to be a continuous open bijection without requiring that a code for the inverse map exists. It is unknown what effect this change of definition would have on the reverse mathematics results presented in this paper (although it is known that this change would not affect the reversal of MFMT to Π^1_2-CA_0). We feel that our definition of a homeomorphism is a natural choice. A particular advantage of our definition is that it makes the homeomorphism relation symmetric.

2.2 Existence of points and the Baire Category Theorem

In this section, we explore the problem of constructing maximal filters on countable posets. We show in ACA_0 that every basic open neighborhood of a countably based MF space is nonempty. We then give a robust definition of a dense open set in a countably based MF space and show in ACA_0 that the intersection of countably many dense open sets is dense (this is the Baire Category Theorem for countably based MF spaces).

Not all facts about the existence of points can be proved in ACA_0. We end the section by showing that the principle that every filter is contained in a maximal filter is equivalent to Π^1_1-CA_0 over RCA_0.

Lemma 2.1. *The following is provable in* ACA_0. *Let P be a countable poset and let $q \in P$. There is an $x \in \mathrm{MF}(P)$ such that $x \in N_q$. Moreover, there is an arithmetically definable functional $\Phi\colon P \to \mathrm{MF}(P)$ such that $\Phi(q) \in N_q$ for all $q \in P$.*

Proof. Fix an enumeration $\langle p_i \mid i \in \mathbb{N}\rangle$ of P. Begin by using arithmetical comprehension to form the set $A = \{\langle p, q\rangle \in P \times P \mid p \perp q\}$. We construct a descending sequence $\langle q_i \mid i \in \mathbb{N}\rangle$ inductively. At stage 0, let $q_0 = q$. At stage $n+1$, use A to determine if there is a common extension of p_n and q_n. If there is, let q_{n+1} be the first common extension to appear in the enumeration of P. If $p_n \perp q_n$ then let $q_{n+1} = q_n$. (It can be seen that the sequence $\langle q_n \rangle$ is Δ^0_1 relative to A.) Let x be the upward closure of $\{q_i \mid i \in \mathbb{N}\}$. It follows from the construction that x is a maximal filter on P and $q \in x$.

Because the construction of x was uniformly Δ^0_1 definable from q and A, the proof shows that there is an arithmetically definable functional Φ satisfying the final claim of the lemma. □

We now turn to the problem of defining a dense open set of a countably based MF space. We show that three possible definitions are equivalent. The first condition is entirely algebraic, while the second and third are topological.

Lemma 2.2. *Let P be a countable poset and let $U \subseteq P$. ACA_0 proves that the following are equivalent.*

1. *For every $p \in P$ there is a $q \in U$ and an $r \in P$ with $r \preceq p$ and $r \preceq q$ (See Remark 2.4).*
2. *For every point $x \in \mathrm{MF}(P)$ and every open set N_V with $x \in N_V$ there is a point $y \in \mathrm{MF}(P)$ such that $y \in N_U \cap N_V$.*
3. *For every point $x \in \mathrm{MF}(P)$ there is a sequence $\langle y_i \mid i \in \mathbb{N} \rangle$ such that $y_i \in N_U$ for each $i \in \mathbb{N}$ and $\langle y_i \rangle$ is eventually inside every open neighborhood of x.*

Proof. We work in ACA_0 and show that (1) implies (3); the proofs of the remaining implications are straightforward. Let $\Phi\colon P \to \mathrm{MF}(P)$ be as in the statement of Lemma 2.1. Fix $x \in \mathrm{MF}(P)$ and let $\langle p_i \mid i \in \mathbb{N} \rangle$ be a descending sequence such that x is the upward closure of $\{p_i\}$. For each i, we may effectively find $r_i \in P$ such that $r_i \preceq p_i$ and $r_i \preceq q_i$ for some $q_i \in U$. Let $y_i = \Phi(r_i)$, for $i \in \mathbb{N}$. It is easy to show $\langle y_i \rangle$ satisfies the conclusion of (3). □

Remark 2.4. We cannot replace (1) in Lemma 2.2 with the following definition from axiomatic set theory:

4. For every $p \in P$ there is a $q \in U$ with $q \preceq p$.

Although (4) implies (1), the converse implication is disprovable in RCA_0.

Definition 2.6. The following definition is made in ACA_0. Let U be a subset of a countable poset P. We say that U is *dense* if U satisfies any of the equivalent conditions in Lemma 2.2.

We now give an analogue of the Baire Category Theorem for countably based MF spaces. The proof is similar to that of Lemma 2.1.

Lemma 2.3 (Baire Category Theorem)**.** *The following is provable in* ACA_0*. Let* P *be a countable poset and let* $\langle U(i) \mid i \in \mathbb{N}\rangle$ *be a sequence of dense open subsets of* $\mathrm{MF}(P)$*. Let* $p \in P$ *be fixed. Then* $N_p \cap \bigcap_{i\in\mathbb{N}} N_{U(i)}$ *is nonempty.*

In ZFC, Zorn's lemma implies that each filter on a countable poset is included in a maximal filter. We now show that that Π^1_1 comprehension is required to extend arbitrary filters on countable posets to maximal filters.

Lemma 2.4. *The following is provable in* $\Pi^1_1\text{-}\mathsf{CA}_0$*. Every filter on a countable poset extends to a maximal filter.*

Proof. The proof uses countable coded β-models. A *countable coded β-model* is a sequence $M = \langle X_i \mid i \in \mathbb{N}\rangle$ of subsets of $\mathbb{N}$ such that

$$\Phi \Leftrightarrow M \models \Phi$$

holds for each Π^1_1 formula Φ with parameters from M. Here $M \models \Phi$ denotes the satisfaction predicate for M, which can be proven to exist in $\Pi^1_1\text{-}\mathsf{CA}_0$. It is known that $\Pi^1_1\text{-}\mathsf{CA}_0$ proves that for each set $X \subseteq \mathbb{N}$ there is a countable coded β-model containing X. See Section VII.2 of [18] for details.

We work in $\Pi^1_1\text{-}\mathsf{CA}_0$. Let P be a countable poset and let F be a filter on P. Let M be a countable coded β-model containing P and F. Consider the formula

$$\Phi(X, p) \equiv \exists Y(\mathrm{Filt}(Y) \wedge p \in Y \wedge X \subseteq Y),$$

where $\mathrm{Filt}(Y)$ is the canonical arithmetical formula which says that Y is a filter on P. It is clear that Φ is Σ^1_1, and thus Φ is absolute to M.

Let $\langle p_i \mid i \in \mathbb{N}\rangle$ be an enumeration of P. We construct a sequence $\langle X_i \mid i \in \mathbb{N}\rangle$ of filters on P inductively. Let $X_0 = X \in M$. At stage $n+1$, assume by induction that $X_n \in M$. Determine whether $\Phi(X_n, p_n)$ holds in M. If it does, then choose $X_{n+1} \in M$ to be a filter on P such that $p_n \in X_{n+1}$ and $X_n \subseteq X_{n+1}$. Otherwise, let $X_{n+1} = X_n$. This completes the construction.

Let $x = \bigcup_{n\in\mathbb{N}} X_n$. It is clear that x is a filter on P, since x is the increasing union of filters on P. Assume that $x \cup \{p_n\}$ extends to a filter on P. Then $\Phi(X_n, p_n)$ holds, so $M \models \Phi(X_n, p_n)$. Thus $p_n \in X_{n+1} \subseteq x$. This shows that x is a maximal filter. □

Theorem 2.4. *The following are equivalent over* RCA_0*.*

1. Π^1_1-CA_0.

2. Every filter on a countable poset extends to a maximal filter.

Proof. Lemma 2.4 shows that Π^1_1-CA_0 proves (2); we must prove the converse in RCA_0. Let $\langle T_i \mid i \in \mathbb{N}\rangle$ be a sequence of subtrees of $\mathbb{N}^{<\mathbb{N}}$. We will show that the set $\{i \in \mathbb{N} \mid T_i \text{ has an infinite path}\}$ exists; this suffices to establish Π^1_1-CA_0.

We begin by forming a sequence $\langle P_i\rangle$ of countable posets. For each $i \in \mathbb{N}$, P_i consists of T_i plus infinitely many new elements $\{a^i_j \mid j \in \mathbb{N}\}$. The order $\preceq_i$ on P_i is the smallest order such that:

1. $\sigma \preceq_i \tau$ whenever $\sigma, \tau \in T_i$ and σ extends τ.
2. $\sigma \preceq_i a^i_j$ whenever $\sigma \in T_i$ and $\text{length}(\sigma) \geq j$.

We now define a poset P as follows. An element of P is a sequence $\langle p_0, p_1, \ldots, p_n\rangle$ with $p_i \in P_i$ for $i \leq n$. We order P by setting $\langle p_0, \ldots, p_n\rangle \preceq \langle q_o, \ldots, q_m\rangle$ if $n \geq m$ and $p_i \preceq_i q_i$ for $i \leq m$. It is clear that this poset may be formed in RCA_0. Let

$$F = \{\langle a^0_{i(0)}, a^1_{i(1)}, \ldots, a^n_{i(n)}\rangle \mid i(0), i(1), \ldots, i(n) \in \mathbb{N}\}.$$

Then F is a filter on P. Let G be any extension of F to a maximal filter.

Claim: For each $i \in \mathbb{N}$, let $p_i = \langle a^0_0, a^1_0, \ldots, a^{i-1}_0, \langle\rangle\rangle$. The tree T_i has a path if and only if $p_i \in G$.

First, suppose that $p_i \in G$. For each $k \in \mathbb{N}$ the element $\langle a^0_0, \ldots, a^{i-1}_0, a^i_k\rangle$ is in G and thus there is a common extension $\langle a^0_0, \ldots, a^{i-1}_0, \tau_k\rangle$ in G. The sequence $\langle \tau_k \mid k \in \mathbb{N}\rangle$ is linearly ordered in T_i and contains elements of arbitrary length; thus the sequence gives a path through T_i.

Now, suppose that T_i has a path $\langle \tau_k \mid k \in \mathbb{N}\rangle$, where $\text{length}(\tau_k) = k$ for each k. Let

$$G' = \{\langle p_0, \ldots, p_{i-1}, \tau_k, p_{i+1}, \ldots, p_r\rangle \mid k \in \mathbb{N}, \langle p_0, \ldots, p_i, \ldots, p_r\rangle \in G\}.$$

The set G' may be formed by Δ^0_1 comprehension. It is straightforward to show that $G \cup G'$ is a filter on P. Thus $G' \subseteq G$ because G is maximal. Since $\langle a^0_0, \ldots, a^i_0\rangle \in G$, we see that $\langle a^0_0, \ldots, a^{i-1}_0, \langle\rangle\rangle \in G$. This completes the proof of the claim.

In light of the claim, we have the following equality of sets:

$$\{i \in \mathbb{N} \mid T_i \text{ has a path}\} = \{i \in \mathbb{N} \mid \langle a^0_0, a^1_0, \ldots, a^{i-1}_0, \langle\rangle\rangle \in G\}$$

This set can be formed by Δ^0_1 comprehension relative to G, using the definition on the right. □

We draw the following computable counterexample from the previous proof.

Corollary 2.1. *There is a computable poset P and a computable filter $F \subseteq P$ such that any maximal filter G extending F is Σ^1_1 complete.*

2.3 The strong Choquet property

In this section, we work in ZFC. We show that all MF spaces have the strong Choquet property (see Definition 2.7). A classical result of Choquet shows that all metrizable spaces with the strong Choquet property are completely metrizable. Thus all metrizable countably based MF spaces are homeomorphic to complete separable metric spaces (Corollary 2.2). A proof will appear in [15] that a topological space is homeomorphic to a countably based MF space if and only if it is second-countable, T_1, and has the strong Choquet property. It will also be shown there that not all T_1 spaces with the strong Choquet property are homeomorphic to MF spaces.

Definition 2.7. The *strong Choquet game* on a topological space X is a certain Gale–Stewart game defined as follows. At stage n, player I plays a pair $\langle U_n, x_n \rangle$ such that U_n is an open subset of X and $x_n \in U_n$. Player II responds by playing an open set V_n such that $x_n \in V_n \subseteq U_n$. At stage $n+1$, player I is required to choose U_{n+1} such that $U_{n+1} \subseteq V_n$ (but player I is not required to choose U_{n+1} such that $x_n \in U_{n+1}$). Player I wins the game if $\bigcap_i U_i$ is empty (or, equivalently, if $\bigcap_i V_i$ is empty). Otherwise, player II wins.

A topological space X has the *strong Choquet property* if player II has a winning strategy for the strong Choquet game on X. A proof of Choquet's metrization theorem, which states that a metric space is completely metrizable if and only if it has the strong Choquet property, may be found in Section 8.D of [5]. We remark that Martin has shown [11] that any space consisting of the maximal points on a domain, with the Scott topology, has the strong Choquet property.

Theorem 2.5. *For every poset P, the space* $\mathrm{MF}(P)$ *has the strong Choquet property.*

Proof. We informally describe a winning strategy for player II in the strong Choquet game on $\mathrm{MF}(P)$. In addition to making the required moves, player II will build an auxiliary sequence $\langle p(i) \in P \mid i \in \mathbb{N} \rangle$ such that

$p(i+1) \preceq p(i)$ for each $i \in \mathbb{N}$. At stage n of the game, player II will define $p(n)$.

At stage 0, player I plays an open set U_0 and a point $x_0 \in U_0$. Player II chooses $p(0) \in P$ such that $x_0 \in N_{p(0)}$ and $N_{p(0)} \subseteq U$. Then player II plays $N_{p(0)}$.

At stage $n+1$, player I plays a point x_{n+1} and an open set U_{n+1} with $x_{n+1} \in U_{n+1}$. By induction and the rules of the game, $U_{n+1} \subseteq N_{p(n)}$. Thus $x_{n+1} \in N_{p(n)}$. Player II chooses $p(n+1)$ such that $p(n+1) \preceq p(n)$, $x_{n+1} \in N_{p(n+1)}$, and $N_{p(n+1)} \subseteq U_{n+1}$. Then player II plays $N_{p(n+1)}$.

We now show that player II will win each play of the strong Choquet game that follows this strategy. By induction, we have the inclusions

$$U_0 \supseteq N_{p(0)} \supseteq U_1 \supseteq N_{p(1)} \supseteq U_2 \supseteq \cdots .$$

Let $x \in \mathrm{MF}(P)$ be any maximal filter extending the set $\{p(i) \mid i \in \mathbb{N}\}$. Clearly $x \in \bigcap_i N_{p(i)}$, so this intersection is nonempty. Thus player II has won the strong Choquet game. $\square$

The next corollary follows immediately from the Theorem 2.5 and Choquet's metrization theorem.

Corollary 2.2. *Any metrizable MF space is completely metrizable. Any countably based metrizable MF space is homeomorphic to a complete separable metric space.*

Remark 2.5. The topological product of any collection of strong Choquet spaces is again a strong Choquet space. Some well-known subspaces of these product topologies are not strong Choquet spaces, however. If Y is an infinite-dimensional separable Banach space then the space of bounded linear functionals on Y with the weak-$*$ topology does not have the strong Choquet property. The set of continuous real-valued functions on the unit interval with the topology of pointwise convergence does not have the strong Choquet property. Thus these spaces are not homeomorphic to MF spaces.

2.4 Closure properties of the class of MF spaces

In this section, we show that the class of countably based MF spaces is closed under taking countable topological products and G_δ subspaces. Products may be formed in ACA_0, but Π^1_1-CA_0 is required to prove that every nonempty G_δ subspace of a countably based MF space is homeomorphic to a countably based MF space.

Theorem 2.6. *The following is provable in* ZFC*. Let* $\langle P_i \rangle$ *be a sequence of countable posets. There is a poset* P *such that* $\mathrm{MF}(P)$ *is homeomorphic to the product topology* $\prod_i P_i$*.*

Proof. We may assume that each poset P_i has a maximal element. We define P to consist of finite sequences $\langle p_0, \ldots, p_i \rangle$ such that $p_i \in P_i$ for each $i \leq n$. We let $\langle p_0, \ldots, p_n \rangle \preceq \langle p'_0, \ldots, p'_m \rangle$ if $n \geq m$ and $p_i \preceq p'_i$ in P_i for each $i \leq m$.

For each $i \in \mathbb{N}$ we define a partial projection function $\pi_i \colon P \to P_i$ by letting $\pi_i(\langle p_0, \ldots, p_n \rangle) = p_i$ if $i \leq n$. We extend each π_i to $\mathrm{MF}(P)$ by letting $\pi_i(F) = \{\pi_i(p) \mid p \in F\}$ for $F \in \mathrm{MF}(P)$.

It is straightforward to show that a filter F on P is maximal if and only if $\pi_i(F)$ is maximal on P_i for every $i \in \mathbb{N}$. □

We use the construction just given to define countable products of MF spaces in second-order arithmetic. It is clear that if $\langle P_i \rangle$ is a sequence of countable posets then the poset P constructed in Theorem 2.6, and a sequence of codes for the partial projection functions, may be formed in ACA_0. Thus, working in ACA_0, we may define the product of $\langle \mathrm{MF}(P_i) \rangle$ to be $\mathrm{MF}(P)$ for this poset. We will not make further use of this definition, however.

Lemma 2.5. *The following is provable in* Π^1_1-CA_0*. Let* P *be a countable poset and let* $\langle U_i \mid i \in \mathbb{N} \rangle$ *be a sequence of open sets in* $\mathrm{MF}(P)$ *such that* $\bigcap_i U_i$ *is nonempty. There is a countable poset* Q *such that* $\mathrm{MF}(Q) \cong \bigcap_i U_i$*.*

Proof. We may assume that P has no minimal elements by adjoining an infinite descending sequence below each minimal element. Let $\langle U_i \mid i \in \mathbb{N} \rangle$ be a sequence of open sets such that $U = \bigcap_i U_i$ is a nonempty G_δ subspace of $\mathrm{MF}(P)$.

We define Q to be the set of pairs $\langle n, p \rangle \in \mathbb{N} \times P$ such that $n > 0$, $p \in P$, $N_p \subseteq \bigcap_{j=0}^n U_j$, and $N_p \cap U \neq \varnothing$. We let $\langle n, p \rangle \prec_Q \langle m, q \rangle$ if $m < n$ and $p \prec q$. The following explicit definition shows that Q may be formed in Π^1_1-CA_0:

$$\begin{aligned} Q = \{\langle n, p \rangle \mid\, & p \in P \wedge \forall i \leq n \, [N_p \subseteq U_i] \\ & \wedge \exists F \subseteq P \, [\mathrm{LO}(F) \wedge p \in F \wedge \forall i \, \exists q \in F \exists r \in U_i \, (q \preceq r)]\}, \end{aligned}$$

where $\mathrm{LO}(F)$ is the canonical formula asserting that F is linearly ordered. The definition of Q quantifies over descending sequences which meet each open set U_i rather than quantifying over maximal filters; this method allows us to avoid Π^1_2 comprehension. We are using the fact that, over Π^1_1-CA_0, a G_δ set $U = \bigcap_i U_i$ contains a point of $\mathrm{MF}(P)$ if and only if there is a linearly ordered $F \subseteq P$ that meets each open set U_i (see Lemma 2.4).

Given $\langle n, p\rangle \in Q$, we may choose a point $x \in N_p \cap U$. Because $x \in U_{n+1}$, there is some q such that $x \in N_q \subseteq U_{n+1}$. Thus, if we let r be a common extension of p and q in x then $\langle n+1, r\rangle \in Q$ and $\langle n+1, r\rangle \prec_Q \langle n, p\rangle$. This shows that Q has no minimal elements.

Suppose that we are given a point $x \in U$. We define $F(x) \subseteq Q$ by

$$F(x) = \{\langle n, p\rangle \in Q \mid x \in N_p\}.$$

We now show that $F(x)$ is a maximal filter. Suppose that there is some $\langle n, p\rangle \in Q$ such that $F(x) \cup \{\langle n, p\rangle\}$ extends to a filter G. Let $G' \subseteq P$ be the set of elements of P that appear in G. Clearly $x \subseteq G'$; since x is maximal, this means $x = G'$. Thus $x \in N_p$. We conclude $F(x)$ is maximal.

The map $F\colon U \to \mathrm{MF}(Q)$ is represented by the code

$$\{\langle 0, p, \langle n, p\rangle\rangle \mid \langle n, p\rangle \in Q\},$$

which may be formed by Δ^0_1 comprehension relative to Q. The inverse to this map is encoded by $\{\langle 0, \langle n, p\rangle, p\rangle \mid \langle n, p\rangle \in Q\}$.

Claim: F is surjective. Let G be any maximal filter on Q, and G' be the set of elements of P that appear in the conditions in G. We will show that the upward closure of G' is a maximal filter on P and the point represented by G' is in U.

It is straightforward to show that the upward closure of G' is a filter on P. Since there are no minimal elements in Q, the numbers n appearing in the conditions in G must be arbitrarily large; hence any point in the G_δ set coded by G' must be in U.

To finish our proof of the claim, we need to show that the upward closure of G' is a maximal filter. Suppose that $p \in P$ and $G' \cup \{p\}$ extends to a maximal filter x on P. This implies $G \subseteq F(x)$. Because G is maximal, $G = F(x)$. Thus $p \in G$. This completes the proof of the claim.

It only remains to check that F is a homeomorphism; this follows from the fact that $x \in p \Leftrightarrow \langle n, p\rangle \in F(x)$ whenever $\langle n, p\rangle \in Q$. □

Theorem 2.7. *The following are equivalent over* ACA_0.

1. Π^1_1-CA_0.

2. *Let P be a countable poset and let $\langle U_i \mid i \in \mathbb{N}\rangle$ be a sequence of open sets in* $\mathrm{MF}(P)$ *such that $\bigcap_i U_i$ is nonempty. There is a countable poset Q such that* $\mathrm{MF}(Q) \cong \bigcap_i U(i)$.

Proof. The implication (1)⇒(2) is given by Lemma 2.5; we prove the converse implication here. We work in ACA_0 and show that (2) implies every closed subset of a complete separable metric space has a countable dense subset. Brown has shown [1] that this implies Π^1_1-CA_0 over RCA_0.

Let $\widehat{A}$ be a complete separable metric space and let C be a closed subset of $\widehat{A}$. It is provable in RCA_0 that every closed subset of a complete separable metric space is a G_δ set; let $C = \bigcap_i V_i$, where $\langle V_i \mid i \in \mathbb{N}\rangle$ is a sequence of coded open subsets of $\widehat{A}$.

Working in ACA_0, we may form the canonical poset P such that $\widehat{A} \cong \mathrm{MF}(P)$ (see Theorem 2.1). We may also convert each open set V_i to a coded open subset U_i of $\mathrm{MF}(P)$ such that $V_i \cong U_i$ for each $i \in \mathbb{N}$.

We apply (2) to find a poset Q such that $\mathrm{MF}(Q) \cong \bigcap_i U_i$ (note $\bigcap_i U_i \cong \bigcap_i V_i = C$). ACA_0 can construct a countable dense subset D of $\mathrm{MF}(Q)$. By applying the homeomorphisms, we may convert D to a countable dense subset of $\bigcap_i V_i$. □

2.5 A representation of coanalytic sets

A subset of $\mathbb{N}^\mathbb{N}$ is *coanalytic* if it is definable by a Π^1_1 formula (which may have set parameters). Our goal in this section is to prove that coanalytic subsets of $\mathbb{N}^\mathbb{N}$ may be represented as closed subsets of countably based MF spaces. This representation will play an important role in reversals in Section 4.

A countably based MF space is *regular* if for each point x and any neighborhood N_p of x there is a neighborhood N_q of x such that the closure of N_q is a subset of N_p.

Theorem 2.8. *The following is provable in ACA_0. Let S be a coanalytic subset of $\mathbb{N}^\mathbb{N}$. There is a countably based Hausdorff MF space X with a closed set C such that $C \cong S$. (The homeomorphism is represented as a canonically arithmetically definable functional from S to C with a canonically arithmetically definable inverse.) Moreover, if S is finite then X is a regular space.*

Proof. Let $\Psi(X)$ be a Π^1_1 formula with one free set variable such that $S = \{X \mid \Psi(X)\}$. There may be set parameters in Ψ. Write Ψ in normal form: there is a Δ^0_0 formula ρ such that $\Psi(X) \Leftrightarrow \forall Y\, \exists n\, \rho(X[n], Y[n])$ holds for all $X \in \mathbb{N}^\mathbb{N}$. Let

$$P = \{\langle\sigma\rangle \mid \sigma \in \mathbb{N}^{<\mathbb{N}}\} \cup \{\langle\sigma,\tau\rangle \mid \sigma,\tau \in \mathbb{N}^{<\mathbb{N}} \wedge |\sigma| = |\tau| \wedge \rho(\sigma,\tau)\}.$$

The order on P is smallest order containing the following:

1. $\langle\sigma\rangle \preceq \langle\sigma'\rangle$ for all $\sigma' \subseteq \sigma$,
2. $\langle\sigma,\tau\rangle \preceq \langle\sigma',\tau'\rangle$ for all $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$,
3. $\langle\sigma,\tau\rangle \preceq \langle\sigma'\rangle$ for all $\sigma' \subseteq \sigma$.

We will now partition $\mathrm{MF}(P)$ into three disjoint classes of maximal filters. *Class 1* consists of the principal maximal filters on P. *Class 2* consists of the nonprincipal maximal filters which contain an element of P of the form $\langle\sigma,\tau\rangle$. *Class 3* consists of the nonprincipal maximal filters which do not contain an element of the form $\langle\sigma,\tau\rangle$. It is clear that this list is exhaustive. The filters in class 1 are generated by minimal elements of P, while all other filters contain an infinite strictly descending sequence. If F is a filter in class 2 then there must be $X_F, Y_F \in \mathbb{N}^\mathbb{N}$ such that $F = \{\langle X_F[m], Y_F[m]\rangle \mid m \in \mathbb{N}\}$; clearly $\forall m\, \rho(X_F[m], Y_F[m])$ holds and thus $\Psi(X_F)$ is false. A filter G in class 3 has no minimal element but does not contain a condition of the form $\langle\sigma,\tau\rangle$. Thus every condition in G is of the form $\langle\sigma\rangle$ and there is an $X_G \in \mathbb{N}^\mathbb{N}$ such that $G = \{\langle X_G[m]\rangle \mid m \in \mathbb{N}\}$. Because G is maximal, there must not be a $Y \in \mathbb{N}^\mathbb{N}$ such that $\forall m\, \rho(X_G[m], Y[m])$ holds. Thus $\Psi(X_G)$ holds. This shows that the filters in class 3 are in correspondence with the elements of $\{X \mid \Psi(X)\}$. The subspace topology of $\mathrm{MF}(P)$ on the set of maximal filters of class 3 is clearly the same as the Baire topology.

The set of all the filters of class 3 is closed in $\mathrm{MF}(P)$, because it is the complement of the coded open set $\bigcup\{N_{\langle\sigma,\tau\rangle} \mid \langle\sigma,\tau\rangle \in P\}$.

The proof that $\mathrm{MF}(P)$ is Hausdorff requires several cases. We prove the only nontrivial case. Suppose that $F \in \mathrm{MF}(P)$ is a filter of class 2 and $G \in \mathrm{MF}(P)$ is of class 3. As above, let $F = \{\langle X_F[m], Y_F[m]\rangle\}$ and let $G = \{\langle X_G[m]\rangle\}$. Since $F \neq G$, $X_F \neq X_G$. Thus for some m we have $X_G[m] \perp X_F[m]$. Thus $G \in N_{\langle X_G[m]\rangle}$, $F \in N_{\langle X_F[m]\rangle}$, and $N_{X_G[m]} \cap N_{X_F[m]} = \varnothing$.

Now assume that $\{X \mid \Psi(X)\}$ is finite. We show that $\mathrm{MF}(P)$ is regular. Because $\mathrm{MF}(P)$ satisfies the T_1 axiom, we only need to show that $\mathrm{MF}(P)$ satisfies the T_3 axiom. That is, given a filter $F \in \mathrm{MF}(P)$ and a neighborhood N_p of F we must show that there is a neighborhood N_q of F such that $\mathrm{cl}(N_q) \subseteq N_p$. The proof divides into three cases, depending on the class of filter that F belongs to.

If F is in class 1 then we may take q to be the minimal element of P which generates F; because $N_q = \{F\}$, we have $N_q = \mathrm{cl}(N_q)$.

If F is in class 2, and there is no filter of class 3 in N_p, then N_p is closed. If there is a filter G of class 3 in N_p, then it may happen that $G \in \mathrm{cl}(N_p)$. But because G is in class 3, there must be a condition $q = \langle\sigma,\tau\rangle \in F$, with

$q < p$, and a condition $\langle \sigma' \rangle \in G$ such that $\sigma \perp \sigma'$; otherwise $G \subseteq F$, which is impossible. By extending σ, σ' if necessary, we can ensure that there is no $G' \in C$ at all with $\langle \sigma \rangle \in G'$. This is because C is finite. Thus $C \cap \mathrm{cl}(N_q) = \varnothing$, which means that any filter in the closure of N_q is of class 1 or class 2. It is clear that a filter of class 1 or 2 in the closure of N_q is actually in N_q; thus $F \in \mathrm{cl}(N_q) \subseteq N_p$.

If F is in class 3, then p is of the form $\langle \sigma \rangle$. We claim that N_p is closed. Let G be a filter not in N_p. If G is of class 1 then G is isolated and thus $G \notin \mathrm{cl}(N_p)$. If G is in class 2 then G must be in a neighborhood of the form $\langle \sigma' \rangle$ with $\sigma \perp \sigma'$; thus $G \notin \mathrm{cl}(N_p)$. If G is of class 3 then there is a σ' such that $G' \in N_{\langle \sigma' \rangle}$ and $\sigma \perp \sigma'$; thus $G' \notin \mathrm{cl}(N_p)$. This shows that N_p is closed. $\square$

3 Cardinality of countably based MF spaces

Definition 3.1. A nonempty closed subset of a countably based MF space is *perfect* if it has no isolated points in the subspace topology. A definable subset U of a countably based MF space X *contains a perfect set* if there is a perfect closed subset C of X such that $C \subseteq U$.

We first prove an analogue of the classical theorem that an injective continuous image of the space $2^{\mathbb{N}}$ is a perfect closed set.

Definition 3.2. The following definition is made in RCA_0. Let P and Q be countable posets and let ϕ be a function from P to Q. We say ϕ is *order preserving* if the following hold for all p, $p' \in P$.

1. $p \preceq_P p'$ if and only if $\phi(p) \preceq_Q \phi(p')$.
2. $p \perp_P p'$ if and only if $\phi(p) \perp_Q \phi(p')$.

For each filter $F \subseteq P$, we write $\phi(F)$ for $\{\phi(p) \mid p \in F\}$.

Note that an order-preserving map must send filters to filters but may send maximal filters to nonmaximal filters.

Lemma 3.1. *The following is provable in* ACA_0. *Let P be a countable poset and let U be a definable subset of* $\mathrm{MF}(P)$. *If there is an order-preserving* $\phi\colon 2^{<\mathbb{N}} \to P$ *such that $\phi(F)$ is maximal and $\phi(F) \in U$ for all $F \in 2^{\mathbb{N}}$ then U contains a perfect subset.*

Proof. The function ϕ may be recognized as a code for a continuous function Φ from $2^{\mathbb{N}}$ to U. Because ϕ is order preserving, Φ is injective. It is not hard to see that the range of Φ has no isolated points. Thus we only need to show that the range of Φ is closed, that is, that there is a coded open subset of $\mathrm{MF}(P)$ complementary to the range of Φ.

Using arithmetical comprehension, we may form the coded open set

$$V = \{q \in P \mid \neg\exists\sigma \in 2^{<\mathbb{N}}[\phi(\sigma) \preceq q]\}.$$

It is clear that V is disjoint from the range of Φ. We will show that V is complementary to the range of Φ, which completes the proof.

Let $y \in \mathrm{MF}(P) \setminus V$ be fixed. Thus for every neighborhood N_p of y there is a $\sigma \in 2^{<\mathbb{N}}$ with $\phi(\sigma) \preceq p$. We show that y is in the range of Φ (this is, essentially, because $2^{\mathbb{N}}$ is compact). Fix a descending sequence $\langle p_i \rangle$ such that $y = \mathrm{ucl}\{p_i\}$. We construct a sequence $\langle \sigma_i \rangle$ inductively. Choose σ_0 such that $\phi(\sigma_0) \preceq p_0$. Given σ_i, choose τ such that $\phi(\tau) \preceq p_{i+1}$. Because ϕ is order preserving, τ and σ_i are compatible, and we can choose σ_{i+1} such that $\sigma_{i+1} \preceq p_{i+1}$ and $|\sigma_{i+1}| > i + 1$. This construction ensures that the set $\{\sigma_i\}$ determines an element of $2^{\mathbb{N}}$ which maps to y. □

Working in ACA_0, we say that $\mathrm{MF}(P)$ is *countable* if there is a sequence $\langle X_i \subseteq \mathbb{N} \mid i \in \mathbb{N}\rangle$ of maximal filters such that for every $F \in \mathrm{MF}(P)$ there is an $i \in \mathbb{N}$ such that $F = X_i$. We remark that ACA_0 proves that we may refine such a sequence so that each maximal filter appears exactly once.

Theorem 3.1. *The following is provable in Π^1_2-CA_0. Let P be a countable poset. If $\mathrm{MF}(P)$ is Hausdorff then exactly one of the following statements holds.*

1. $\mathrm{MF}(P)$ *is countable.*
2. *There is an order-preserving map from $2^{<\mathbb{N}}$ to P.*

Proof. Let P be a countable poset. We define the *poset star game on P*, which is a certain Gale–Stewart game. (The name of the game comes from the fact that it resembles the $*$-game from descriptive set theory; see Section 21.A of [5]). At stage 0, player I chooses p^0_0, $p^0_1 \in P$ and then player II chooses $n(0) \in \{0, 1\}$. At stage $i + 1$, player I chooses p^{i+1}_0, $p^{i+1}_1 \in P$ such that $p^{i+1}_k \preceq p^i_{n(i)}$ for $k \in \{0, 1\}$. Player II chooses $n(i + 1) \in \{0, 1\}$. The play continues this way for $\mathbb{N}$ rounds.

Player I wins if $p^n_0 \perp p^n_1$ for all $n \in \mathbb{N}$, and player II wins otherwise. The set of winning plays for player I is a closed set, and thus for each P either player I or player II has a winning strategy for the game.

A winning strategy for player I immediately gives an order-preserving map from $2^{<\mathbb{N}}$ to P. We will show that if player II has a winning strategy then $\mathrm{MF}(P)$ is countable.

A *position* is a sequence $\langle\langle p_0^0, p_1^0\rangle, \ldots, \langle p_0^l, p_1^l\rangle\rangle$ such that $p_0^i \perp p_1^i$ for $i \le l$ and π respects that strategy s_{II} in the sense that $p_k^{i+1} \preceq p^i_{s_{\mathrm{II}}(\pi[i])}$ for $i \le l$ and $k \in \{0,1\}$. A position is π *consistent* with $x \in \mathrm{MF}(P)$ if $x \in N_{s_{\mathrm{II}}(\pi)}$. Note that there is a Π^0_1 formula which tells if a position π is consistent with x. A position π is a *maximal* position for $x \in \mathrm{MF}(P)$ if π is consistent with x and π cannot be extended to a longer position consistent with x.

Under the assumption that player II has a winning strategy, every point $x \in \mathrm{MF}(P)$ has a maximal position. Otherwise, there is a point x such that every position consistent with x can be extended to a longer position consistent with x. In this case, player I would have an arithmetically definable winning strategy for the poset star game on P, which is impossible.

No position can be maximal for two points $x, y \in \mathrm{MF}(P)$. To see this, assume a position π is consistent with x and y. Because $\mathrm{MF}(P)$ is Hausdorff, there are disjoint basic open neighborhoods p and q of x and y such that player I can play $\langle p, q\rangle$ at the next stage of the poset star game after π. The response of player II will extend π to a position consistent with either x or y.

We may use Π^1_2 comprehension to form the set M of all positions which are maximal for some point. We may then use Π^1_1 choice (see Theorem VII.6.9 of [18]) to form a sequence $\langle x_\pi \mid \pi \in M\rangle$ such that π is maximal for x_π. The arguments in the previous two paragraphs show that every point of $\mathrm{MF}(P)$ is included in the sequence. Thus $\mathrm{MF}(P)$ is countable. □

Corollary 3.1. *The following is provable in* ZFC. *Every countably based Hausdorff MF space is either countable or contains a perfect set.*

Proof. We work in ZFC, which proves that every order-preserving map from $2^{<\mathbb{N}}$ to P can be extended to an injection (not necessarily continuous) from $2^{\mathbb{N}}$ to $\mathrm{MF}(P)$. Thus every countably based MF space is either countable or has cardinality $2^{\aleph_0}$. Every second-countable space of cardinality $2^{\aleph_0}$ has a perfect closed subset (the complement of the union of those basic open sets of cardinality less than $2^{\aleph_0}$). □

Theorem 3.2. *The following statement implies* ATR_0 *over* ACA_0. *If P is a countable poset and $\mathrm{MF}(P)$ is Hausdorff then either $\mathrm{MF}(P)$ is countable or there is an order preserving map from $2^{<\mathbb{N}}$ to P.*

Proof. We will show that the statement implies the perfect set theorem for closed subsets of $2^{\mathbb{N}}$, which in turn implies ATR_0 over RCA_0. It is provable in RCA_0 that every closed subset of $2^{\mathbb{N}}$ arises as the set of paths through a subtree of $2^{<\mathbb{N}}$.

Let T be a subtree of $2^{<\mathbb{N}}$ and regard T as a countable poset, so every maximal filter arises either from a path through T or a dead end of T. By assumption, either $\mathrm{MF}(T)$ is countable or there is an order-preserving map from $2^{<\mathbb{N}}$ to T. This trivially implies that either T has countably many paths or T has a perfect subtree. □

Question 3.1. Determine the reverse mathematics strength of the following statement. Every Hausdorff countably based MF space is either countable or contains a perfect closed set. We have shown this statement is provable in Π^1_2-CA_0 and implies ATR_0.

We now turn to consider closed subspaces of countably based MF spaces; we will use the representation of coanalytic subsets described in Section 2.5.

A definable subset U of a countably based MF space *contains a perfect set in the Baire topology* if the collection of characteristic functions of the maximal filters in U contains a perfect closed subset in $2^{\mathbb{N}}$. This is easily seen to be equivalent to the existence of a perfect subtree T of $2^{<\mathbb{N}}$ such that the paths through T are all characteristic functions of maximal filters in U.

Lemma 3.2. *The following is provable in* ACA_0. *Let U be a definable subset of a countably based Hausdorff MF space. If U contains a perfect subset in the Baire topology then U contains a perfect subset.*

Proof. Let $\langle p_i \rangle$ be the standard enumeration of P. Let T code a perfect subset of U in the Baire topology. Only countably many $f \in [T]$ may have the property that there is an $k \in \mathbb{N}$ such $f(k) = 0$ that for all $n > k$; each such f corresponds to a minimal element of the poset. We may effectively find a perfect subtree of T such that no path through the subtree this property. We replace T by such a perfect subtree if necessary.

We construct an order-preserving map $h\colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ inductively. Let h send the empty sequence to itself. Suppose h is defined on $\sigma \in 2^{<\mathbb{N}}$. We may canonically choose distinct paths $f_0, f_1 \in [T]$ such that $\sigma \subseteq f_m$ for $m = 0, 1$. Because f_0 and f_1 are distinct paths through T, they correspond to distinct filters in X. Because X is Hausdorff, there are $k_0, k_1 > |\sigma|$ such that $f_0(k_0) = 1$, $f_1(k_1) = 1$, and the poset elements p, p' represented by k_0 and k_1 respectively are incompatible. Let $h(\sigma \frown \langle m \rangle) = k_m$ for $m = 0, 1$.

This completes the construction of h. It is straightforward to verify that the map $\phi\colon \sigma \mapsto p_{|h(\sigma)|}$ is an order-preserving map from $2^{<\mathbb{N}}$ to P and $\phi(f) \in U$ for all $f \in 2^{\mathbb{N}}$. Thus, by Lemma 3.1, U has a perfect subset. $\square$

Lemma 3.3. *The following are equivalent over* ACA_0.

1. *Every coanalytic subset of* $\mathbb{N}^{\mathbb{N}}$ *is either countable or contains a perfect closed set.*

2. *Every closed subset of a countably based Hausdorff MF space is either countable or contains a perfect closed set.*

Proof. We first prove that (1) implies (2). Let C be a closed subset of a Hausdorff MF space $\mathrm{MF}(P)$, where P is countable. There is a Π^1_1 formula which defines C as a subset of $\mathbb{N}^{\mathbb{N}}$. Thus (1) implies that C is either countable or C contains a perfect subset in the Baire topology. In the latter case, Lemma 3.2 implies C contains a perfect subset.

To show that (2) implies (1), let U be a coanalytic subset of $\mathbb{N}^{\mathbb{N}}$. Apply Theorem 2.8 to find a countable poset P such that $\mathrm{MF}(P)$ is Hausdorff and there is a closed set $C \subseteq \mathrm{MF}(P)$ with $C \cong U$. Statement (1) follows immediately. $\square$

It is well known that Π^1_1-CA_0 is able to formalize many facts about Gödel's constructible universe L. In particular, there is a sentence $\Phi(A)$ in the language of second-order arithmetic which says that $\aleph_1^{L(A)}$ is countable. The proposition that $\aleph_1^{L(A)}$ is countable for all $A \subseteq \mathbb{N}$ is well known to be independent of ZFC (see Chapter 6 of [9] and Theorem 25.38 of [4]).

Lemma 3.4. *The following are equivalent over* Π^1_1-CA_0.

1. $\aleph_1^{L(A)}$ *is countable for all* $A \subseteq \mathbb{N}$.

2. *Every coanalytic subset of* $\mathbb{N}^{\mathbb{N}}$ *is countable or has a perfect subset.*

Proof. The proof in ZFC given by Mansfield and Weitkamp in Chapter 6 of [9] may be formalized in Π^1_1-CA_0. The proof relies only on properties of the constructible hierarchy that are provable in Π^1_1-CA_0 (see Section VII.4 of [18]). In particular, Simpson has shown that Kondo's Π^1_1 uniformization theorem and Shoenfield's absoluteness theorem are provable in Π^1_1-CA_0; see Theorems VI.2.6 and VII.4.14 of [18]. $\square$

The next theorem follows immediately from Lemmas 3.3 and 3.4.

Theorem 3.3. *The following are equivalent over* Π^1_1-CA_0.

1. *Every closed subset of a countably based Hausdorff MF space is either countable or contains a perfect closed set.*
2. $\aleph_1^{L(A)}$ *is countable for all* $A \subseteq \mathbb{N}$.

Corollary 3.2. *The proposition that every closed subset of a countably based Hausdorff MF space is either countable or contains a perfect closed set is independent of* ZFC *and is false if* $V = L$.

The next corollary follows immediately from Theorem 3.3 and Corollary 3.1.

Corollary 3.3. *The proposition that every closed subset of a countably based MF space is homeomorphic to a countably based Hausdorff MF space is not provable in* ZFC. *In particular, the proposition is false if* $V = L$.

4 Metrizability of countably based MF spaces

In this section, we prove the following principle is equivalent to Π^1_2-CA_0.

> MFMT: A countably based MF space is homeomorphic to a complete separable metric space if and only if it is regular.

The proof of MFMT in Π^1_2-CA_0 is given in Section 4.1; the reversal is established in Section 4.2.

4.1 Metrizability of regular MF spaces

The central goal of this section is to prove MFMT in Π^1_2-CA_0. Our proof proceeds in two stages. In the first stage, we show that any regular countably based MF space X is metrizable. Our construction of the metric is inspired by a result of Schröder in effective topology [17] and by the classical proof of Urysohn's metrization theorem. We do not know whether the metric we have constructed on X is a complete metric. In the second stage of the proof, we construct another metric on X which is complete and induces the same topology as the first metric. Our construction of the complete metric is inspired by the proof of Kechris in Section 8.D of [5] that a metrizable strong Choquet space is completely metrizable. Both that proof and our proof of MFMT proceed by showing that X is a G_δ subset of the complete separable metric space $\langle \hat{A}, d \rangle$, where A is a countable dense subset of X.

The next definition is inspired by the definition of an effectively regular space given by Schröder [17].

Definition 4.1. The following definition is made in ACA_0. Let $X = \mathrm{MF}(P)$ be a countably based MF space. We say that X is *strongly regular* if there is a sequence $\langle R_p \mid p \in P\rangle$ of subsets of P such that $N_p = \bigcup_{q \in R_p} N_q$ for each $p \in P$ and $\mathrm{cl}(N_q) \subseteq N_p$ whenever $q \in R_p$.

Lemma 4.1. *It is provable in Π^1_2-CA_0 that any regular MF space is strongly regular.*

Proof. There is a Π^1_2 formula $\Phi(p, q)$ which holds if and only if $\mathrm{cl}(N_q) \subseteq N_p$. Thus we may define

$$R_p = \{q \in P \mid \mathrm{cl}(N_q) \subseteq N_p\}$$

using Π^1_2 comprehension. The hypothesis that X is regular implies that this sequence $\langle R_p\rangle$ is a witness for the strong regularity of X. □

We will show in Section 4.2 that the statement "every regular countably based MF space is strongly regular" is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

Suppose that X is a metrizable countably based MF space. The next two lemmas consider the problem of converting a code for an open set in the poset topology on X into a code for the same open set in the metric topology on X and the converse problem of converting a code for an open set in the metric topology on X into a code for the same open set in the poset topology on X. Lemma 4.2 shows that codes for an open set in the metric topology may be uniformly converted to codes in the poset topology. Lemma 4.3 shows that codes for open sets in the poset topology may be arithmetically uniformly converted into codes in the metric topology if and only if the space is strongly regular.

Lemma 4.2. *The following is provable in ACA_0. Let X be a metrizable countably based MF space. There is an arithmetically defined functional Φ such that for every $S = \{\langle x_i, r_i\rangle\} \subseteq X \times \mathbb{Q}^+$ we have $\Phi(S) \subseteq P$ and $\bigcup_{\langle x_i, r_i\rangle \in S} B(x_i, r_i) = \bigcup_{p \in \Phi(S)} N_p$.*

Proof. Let d be a metric on X and let A be a countable dense subset of X. For $p \in P$, $x \in X$, and $r \in \mathbb{Q}^+$, we write $p \ll \langle x, r\rangle$ if and only if there is an $a \in A \cap N_p$ and an $s \in \mathbb{Q}^+$ such that $d(a, b) + d(a, x) + s < r$ for all $b \in A \cap N_p$ (this notation is limited to the current proof). The relation $\ll$, viewed as a three-place predicate of p, x, and r, is uniformly arithmetically definable relative to d and A. For each $S = \{\langle x_i, r_i\rangle\} \subseteq X \times \mathbb{Q}^+$ let

$$\Phi(S) = \{p \mid \exists i (p \ll \langle x_i, r_i\rangle)\}.$$

It is clear that Φ is uniformly arithmetically definable.

It remains to show that $\bigcup_{i\in\mathbb{N}} B(x_i, r_i) = \bigcup_{p\in\Phi(S)} N_p$ for each $S = \{\langle x_i, r_i\rangle\} \subseteq X \times \mathbb{Q}^+$. First, suppose that $p \in \Phi(S)$ and $y \in N_p$. Then for some $i \in \mathbb{N}$ we have $p \ll \langle x_i, r_i\rangle$. From the definition of $\ll$ we see that $d(y, x_i) < r$, which means $y \in B(x_i, r_i)$. For the converse, suppose that $y \in B(x_i, r_i)$ for some $i \in \mathbb{N}$. Choose $a \in A$ and $t \in \mathbb{Q}^+$ such that $d(a, y) < t$ and $d(x_i, a) + 2t < r_i$. By the definition of metrizability, there is a $p \in P$ such that $y \in N_p$ and $N_p \subseteq B(a, t)$. We will show that $p \ll \langle x_i, r_i\rangle$. Choose $a' \in A\cap N_p$. Then $d(x_i, a') < r_i$ so $a' \in B(x_i, r_i)$. For $b \in A\cap N_p$, $d(a', b) < t$, whence $d(x_i, a') + d(a', b) + t < r_i$. We conclude $p \ll \langle x_i, r_i\rangle$. □

Lemma 4.3. *The following is provable in* ACA_0. *Let* $X = \mathrm{MF}(P)$ *be a metrizable countably based MF space. The following are equivalent.*

1. *X is strongly regular.*
2. *There is an arithmetically defined functional Ψ such that, for every $U \subseteq P$, $\Psi(U) \subseteq X \times \mathbb{Q}^+$ and $\bigcup_{p\in U} N_p = \bigcup_{\langle x,r\rangle\in\Psi(U)} B(x, r)$.*

Proof. Let d be a metric on X, and let A be a countable dense subset of X.

First, suppose that there is a sequence $\langle R_p \mid p \in P\rangle$ witnessing the strong regularity of X. We define a relation $\ll$ on $(X \times \mathbb{Q}) \times P$ by letting $\langle x, r\rangle \ll p$ if and only if there is a $q \in R_p$ such that $B(x, r) \cap A \subseteq N_q$ (this notation is, again, limited to the current proof). It is clear that $\langle x, r\rangle \ll p$ implies $B(x, r) \subseteq N_p$, because $\mathrm{cl}(B(x, r)) \subseteq \mathrm{cl}(N_q) \subseteq R_p$. Moreover, for every $x \in N_p$ there is a $q \in R_p$ with $x \in N_q$. Thus there is a $y \in X$ and an $r \in \mathbb{Q}^+$ such that $x \in B(y, r) \subseteq N_q$. Choose $a \in A$ and $s \in \mathbb{Q}^+$ such that $d(y, a) + s < r$ and $d(x, a) < s$. Then $x \in B(a, s) \subseteq B(y, r)$. Moreover, $B(y, r) \cap A \subseteq N_q \cap A$, which implies $\langle a, r\rangle \ll p$. Thus $N_p = \bigcup\{B(a, s) \mid \langle a, s\rangle \ll p\}$ for every $p \in P$. For each $U \subseteq P$ we let

$$\Psi(U) = \{\langle a, r\rangle \mid \exists p \in U(\langle a, r\rangle \ll p)\}.$$

It is clear that Ψ is uniformly arithmetically definable, and it follows from the discussion above that

$$\bigcup_{p\in U} N_p = \bigcup_{\langle x,r\rangle\in\Psi(U)} B(x, r).$$

Next, suppose that Ψ is any arithmetically definable functional such that $\bigcup_{p\in U} N_p = \bigcup_{\langle x,r\rangle\in\Psi(U)} B(x, r)$ for each $U \subseteq P$. We wish to show that X is strongly regular. For each $p \in P$, let $V_p = \Psi(\{p\})$. Thus V_p is a sequence of elements of $X \times \mathbb{Q}^+$. Let W_p be the set of all pairs $\langle a, s\rangle \in X \times \mathbb{Q}^+$ such that there exists $\langle x, r\rangle \in V_p$ with $d(a, x) + s < r$. Note that if $\langle a, s\rangle \in T_p$ then $\text{cl}(B(a, s)) \subseteq N_p$, and every $x \in N_p$ is in $B(a, s)$ for some $\langle a, s\rangle \in W_p$. Furthermore, we may uniformly define W_p from p with an arithmetical formula. Let Φ be the functional constructed in Lemma 4.2. For each $p \in P$ let

$$R_p = \{q \in P \mid \exists\langle a, s\rangle \in W_p(q \in \Phi(\{\langle a, s\rangle\}))\}.$$

The set R_p is uniformly arithmetically definable from p, and we may thus form the sequence $\langle R_p\rangle$ in ACA_0. It follows from the construction that $\langle R_p\rangle$ is a witness to the strong regularity of X. □

Definition 4.2. Let P be a countable poset. We define a functional $\perp$ taking subsets of P to subsets of P with the rule

$$U^\perp = \{q \in P \mid \forall p \in U(p \perp q)\}.$$

We often view $\perp$ as a function which takes codes open subsets of $\text{MF}(P)$ to coded open subsets of $\text{MF}(P)$. The next lemma shows that several important properties of $\perp$ are provable in ACA_0; we omit the straightforward proof.

Lemma 4.4. *The following is provable in* ACA_0. *Let* $X = \text{MF}(P)$ *be a countably based MF space and let* U *be a coded open subset of* X. *Then* $X = \text{cl}(U) \cup U^\perp$, $U \cap U^\perp = \varnothing$, $U^\perp = X \setminus \text{cl}(U)$, *and* $U \cap \text{cl}(U^\perp) = \varnothing$.

Lemma 4.5. *The following is provable in* ACA_0. *Let* X *be a countably based strongly regular MF space. There are arithmetically definable functionals* ν_1, ν_2 *with the following property. If* U *and* V *are coded open subsets of* X *and* $\text{cl}(U) \cap \text{cl}(V) = \varnothing$ *then* $\nu_1(U, V)$ *and* $\nu_2(U, V)$ *are disjoint coded open subsets of* X, $\text{cl}(U) \subseteq \nu_1(U, V)$, *and* $\text{cl}(V) \subseteq \nu_2(U, V)$.

Proof. Define an enumeration function $e \colon \mathbb{N} \to P$ such that for every coded open subset U of x we have

$$\{e(U, n)\} = \{q \mid \exists p \in U(q \in R_p)\}.$$

The function e is uniformly arithmetically definable relative to a witness for the strong regularity of X.

For each pair U, V of coded open sets we define a sequence $\langle G_n(U, V) \mid n \in \mathbb{N}\rangle$ of coded open sets

$$G_n(U, V) = \{p \in P \mid p \preceq e(U, n) \wedge \forall i \leq n(p \perp e(V, i))\}.$$

Note that the sequence $\langle G_n(U,V)\rangle$ is uniformly arithmetically definable from U and V. Moreover, for all coded open sets U, V and any $n, m \in \mathbb{N}$, the sets $G_n(U,V)$ and $G_m(V,U)$ code disjoint open subsets of $\mathrm{MF}(P)$.

Define

$$\nu_1(U,V) = \{p \in P \mid \exists n(p \in G_n(V^\perp, U^\perp))\},$$
$$\nu_2(U,V) = \{p \in P \mid \exists n(p \in G_n(U^\perp, V^\perp))\}.$$

It is clear that ν_1 and ν_2 are arithmetically definable functionals.

Now suppose that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \varnothing$. It is clear that $\nu_1(U,V)$ and $\nu_2(U,V)$ must be disjoint. We need to show that $\mathrm{cl}(U) \subseteq \nu_1(U,V)$ and $\mathrm{cl}(V) \subseteq \nu_2(U,V)$. Suppose that $x \in \mathrm{cl}(U)$; then $x \in V^\perp$ and there is an n such that $x \in N_{e(V^\perp,n)}$. Moreover, $x \notin \mathrm{cl}(U^\perp)$, which implies that $x \notin \mathrm{cl}(e(U^\perp, m))$ for all m. Thus there is an $r \in P$ such that $x \in N_r$ and $r \perp e(U^\perp, i)$ for all $i \leq n$. Let p be a common extension of r and $e(V^\perp, n)$ such that $x \in N_p$. Then $p \in G_n(V^\perp, U^\perp)$, whence $x \in G_n(V^\perp, U^\perp) \subseteq \nu_1(U,V)$. This shows $\mathrm{cl}(U) \subseteq \nu_1(U,V)$. Since $\nu_2(U,V) = \nu_1(V,U)$, we have also shown that $\mathrm{cl}(V) \subseteq \nu_2(U,V)$. □

Lemma 4.6. *The following is provable in ACA_0^+ plus Σ^1_1 induction. Let $X = \mathrm{MF}(P)$ be a countably based MF space, let R witness the strong regularity of X, and fix $p \in P$ and $q \in R_p$. Let D be the set of dyadic rational numbers in the interval $[0,1]$. There is a map $g_{p,q}$ from D to coded open subsets of X such $g_{p,q}(0) = \{q\}$, $g_{p,q}(1) = \{p\}$, and $\mathrm{cl}(g_{p,q}(k)) \subseteq g_{p,q}(k')$ whenever $k < k'$.*

Proof. Let ν_1 and ν_2 be as in Lemma 4.5. Note that if V is an open set such that $\mathrm{cl}(V) \subseteq U$ then $\mathrm{cl}(V) \cap \mathrm{cl}(U^\perp) = \varnothing$, for otherwise $U \cap \mathrm{cl}(U^\perp)$ is nonempty.

Each element $d \in D$ can be written in a unique reduced form $a/2^n$ in which n is minimal. We define $\mathrm{length}(d) = n$, the exponent of 2 in the reduced form of d.

We define the map $g_{p,q}$ by induction on length. We write U_k for the coded open set $g_{p,q}(k)$. Let $U_0 = \{q\}$ and $U_1 = \{p\}$. We will maintain the induction hypothesis that if $k < k'$ are in D and $g_{p,q}$ is defined at k and k' then $\mathrm{cl}(U_{k'}) \subseteq U_k$. The induction hypothesis clearly holds for elements of D of length 0. It can be seen that the induction hypothesis is of the form

$$\forall k, k' \forall x((k' < k \wedge x \in \mathrm{MF}(P)) \Rightarrow \Phi(x, k, k'))$$

where Φ is an arithmetical formula which says that $\mathrm{cl}(U_{k'}) \subseteq \mathrm{cl}(U_k)$. Thus Σ^1_1 induction suffices for this argument.

Now assume $g_{p,q}$ is defined for all elements of D of length m and let $k \in D$ have length $m+1$. Write k in reduced form as $a/2^{m+1}$ and let $k_l = (a-1)/2^{m+1}$ and $k_r = (a+1)/2^{m+1}$. Then $k_l < k < k_r$, $\text{length}(k_l) = m$, and length $k_r = m$. Let $U_k = \nu_1(U_{k_l}, U_{k_r}^{\perp})$. By induction, $\text{cl}(U_{k_l}) \subseteq U_k$, and thus $\text{cl}(U_{k_l}) \cap U_{k_r}^{\perp} = \varnothing$ by the argument above. By Lemma 4.5, we know that $\text{cl}(U_{k_l}) \subseteq \nu_1(U_{k_l}, U_{k_r}^{\perp})$, which implies $\text{cl}(U_{k_l}) \subseteq U_k$. Lemma 4.5 also shows that $\text{cl}(U_k)$ is disjoint from $U_{k_r}^{\perp}$; thus $\text{cl}(U_k) \subseteq U_{k_r}$. The induction hypothesis has been maintained.

To complete the proof in ACA_0^+, we note that the construction of $\langle U_k \mid k \in D\rangle$ can be arranged as an induction on $\mathbb{N}$, and the stages of the induction are uniformly given by an arithmetically definable functional. Thus the inductive construction is valid in ACA_0^+. □

Lemma 4.7. *The following is provable in* ACA_0^+ *plus* Σ^1_1 *induction. Let* X *be a strongly regular countably based MF space. There is a sequence* $\langle f_{p,q} \mid p \in P, q \in R_p\rangle$ *of continuous functions from* P *to* $[0,\infty)$ *such that for all* $p \in P$ *and* $q \in R_p$ *we have* $f_{p,q} \upharpoonright N_q = 0$ *and* $f_{p,q} \upharpoonright (X \setminus N_p) = 1$.

Proof. We first note that the construction in Lemma 4.6 was uniform with respect to $p \in P$ and $q \in R_p$. Thus, working in ACA_0^+, we may form a sequence $\langle g_{p,q} \mid p \in P, q \in R_p\rangle$ of functions such that each function $g_{p,q}$ in the sequence satisfies the conclusion of Lemma 4.6. The remainder of the proof will use only ACA_0.

Fix $p \in P$ and $q \in R_p$. We use $g_{p,q}$ to define a real-valued function $f_{p,q} \colon X \to [0,1]$. This definition will be uniform in p and q, allowing us to form the sequence $\langle f_{p,q}\rangle$ from the sequence $\langle g_{p,q}\rangle$. We first extend the domain of $g_{p,q}$ to the set D^+ of all positive dyadic rational numbers by letting $g_{p,q}(k) = X$ for all $k > 1$. We again write U_k for $g_{p,q}(k)$.

We seek to define a code for the function $f_{p,q}$ such that $f_{p,q}(x) = \inf\{k \in D^+ \mid x \in U_k\}$ for all $x \in X$. Clearly this definition of $f_{p,q}$ satisfies the conclusion of the lemma. We only need to show that a code for $f = f_{p,q}$ may be constructed in ACA_0. Note that $f(x) < k$ if and only if there is some $k' < k$ such that $x \in U_{k'}$, if and only if there is some p with $x \in N_p$ and $p \in U_{k'}$. Similarly, $k < f(x)$ if and only if there is a $k'' > k$ with $x \in U_{k''}^{\perp}$, if and only if there is some p with $x \in N_p$ and $p \in U_{k''}^{\perp}$. We use these equivalences to form the code for f, associating $p \in P$ with the interval (a,b) if $a < f(x) < b$ for all $x \in N_p$. □

Lemma 4.8. *The following is provable in* ACA_0^+ *plus* Σ^1_1 *induction. Every strongly regular countably based MF space is metrizable.*

Proof. The proof is parallel to the classical proof of Urysohn's metrization theorem. Let X be a strongly regular countably based MF space. Let $\langle f_{p,q} \mid q \in R_p \rangle$ be the sequence of functions constructed in Lemma 4.7. Reindex these functions as $\langle f_i \mid i \in \mathbb{N} \rangle$. Form a coded continuous function $d\colon X \times X \to [0, \infty)$ such that

$$d(x, y) = \sum_{i \in \mathbb{N}} 2^{-i} |f_i(x) - f_i(y)|.$$

It is straightforward to show that d is a metric on X and that the metric topology induced by d is the same as the poset topology on X. □

The next theorem completes the first stage of our proof of MFMT. The proof follows directly from Lemmas 4.1 and 4.8.

Theorem 4.1. *The following is provable in* Π^1_2-CA_0. *Every regular countably based MF space is metrizable.*

Question 4.1. The reverse mathematics strengths of the following statements are unknown.

1. Every strongly regular countably based MF space is metrizable.
2. Every regular countably based MF space is metrizable.

The next lemma shows in ACA_0 that every G_δ subset of a complete separable metric space is completely metrizable.

Lemma 4.9. *The following is provable in* ACA_0. *If* $\langle U_i \rangle$ *is a sequence of open sets in a complete separable metric space* $\langle \widehat{A}, d \rangle$ *then there is a complete metric* d' *on* $\bigcap_i U_i$ *which gives* $\bigcap_i U_i$ *the same topology as* d. *If* D *is a countable dense subset of* $\bigcap_i U_i$ *then* $\bigcap_i U_i$ *is canonically homeomorphic to the complete separable metric space* $\langle \widehat{D}, d' \rangle$.

Proof. The proof of Theorem 3.11 in [5] shows that a G_δ subspace of a complete separable metric space is completely metrizable. This proof goes through in ACA_0. The rest of the lemma follows easily. □

An open cover of a space is called *point finite* if each point in the space belongs to only finitely many open sets in the cover. A space is paracompact if every open cover of a space can be refined to a point-finite cover. The following technical lemma is the uniformization of Theorem II.7.2 in [18], which shows that RCA_0 proves complete separable metric spaces are paracompact; the proof given in [18] uniformizes without difficulty.

Lemma 4.10. *The following is provable in* RCA_0*. There is an arithmetically defined functional* $\Phi(U, A, d)$ *such that if* $U = \langle U_i \rangle$ *is a sequence of open sets in the complete separable metric space* $\langle \widehat{A}, \hat{d} \rangle$ *then* $\Phi(U, A, d) = \langle V_i \rangle$ *is a sequence of open sets in* $\widehat{A}$*,* $V_i \subseteq U_i$ *for each* i*,* $\bigcup_i V_i = \bigcup_i U_i$*, and each point of* $\widehat{A}$ *is in only finitely many of the sets* V_i*.*

Lemma 4.11. *The following is provable in* ACA_0*. Let* X *be a metrizable countably based MF space. There is a complete separable metric space* $\widehat{A}$ *and a continuous open bijection* h *between* X *and a dense subset of* $\widehat{A}$*. Moreover,* h *is an isometry.*

Proof. Let $X = \mathrm{MF}(P)$ be metrizable with metric d. Let A be a countable dense subset of $\mathrm{MF}(P)$. We will build a code H for a continuous map h from $\mathrm{MF}(P)$ to $\mathrm{MF}(Q)$, where $Q = A \times \mathbb{Q}^+$ is the canonical poset representing the complete separable metric space $\langle \widehat{A}, \widehat{d} \rangle$ (see Theorem 2.1).

For each $p \in P$ we define $\mathrm{diam}(p) \in [0, \infty)$ with the rule

$$\mathrm{diam}(p) = \sup\{d(a, a') \mid a, a' \in N_p \cap A\}.$$

This definition is valid in ACA_0. We let H be the set of all $\langle p, \langle a, r \rangle \rangle$ in $P \times Q$ such that $a \in N_p$ and $\mathrm{diam}(p) < r$.

We first show that $H[x]$ is a filter for each $x \in \mathrm{MF}(P)$. Suppose that $\langle p, \langle a, r \rangle \rangle$ and $\langle q, \langle a', s \rangle \rangle$ are in $H[x]$. Then $d(x, a) < r$ and $d(x, a') < s$. Choose $\epsilon \in \mathbb{Q}^+$ such that $d(x, a) + \epsilon < r$ and $d(x, a') + \epsilon < s$. Choose $u \in P$ such that $x \in N_u$ and $N_u \subseteq B(x, \epsilon/8)$. Note that $\mathrm{diam}(u) \leq \epsilon/4$. Choose $a'' \in A \cap N_u$. Then $\langle u, \langle a'', \epsilon/4 \rangle \rangle \in H$. Now $d(a, a'') + \epsilon/4 < d(a, x) + d(x, a) + \epsilon/4 < d(a, x) + \epsilon/2 < r$, so $B(a'', \epsilon/4)$ is formally included in $B(a, r)$. Similarly $B(a'', \epsilon/4)$ is formally included in $B(a', s)$. This shows that every pair of elements in $H[x]$ have a common extension in $H[x]$.

We next show that $H[x]$ is always maximal. It suffices to show there are $\langle a, r \rangle \in H[x]$ for arbitrarily small r. Fix $x \in \mathrm{MF}(P)$ and let $\langle a_{n(i)} \mid i \in \mathbb{N} \rangle$ be a sequence of points in A which converges to x. Because X is metrizable, for each $m \in \mathbb{N}$ we may choose $p_m \in P$ such that $x \in N_p$ and $\mathrm{diam}(p) < 2^{-(m+1)}$. Choose i large enough that $a_{n_i} \in N_p$. Then $\langle p, \langle a_{n(i)}, 2^m \rangle \rangle \in H$. Since r was arbitrary, there are open balls of arbitrarily small radius in $H[x]$.

We have now shown that H is a code for a continuous function h from $\mathrm{MF}(P)$ to $\mathrm{MF}(Q)$. It is clear that h is an isometry, because the restriction of h to A is an isometry. Because X is metrizable with metric d, every open subset of X in the poset topology is open in the metric topology. Because h is an isometry, we can easily convert a code for an open subset of X in the metric topology to a code for the corresponding open subset of $h(X) \subseteq \widehat{A}$. □

Lemma 4.12. *The following is provable in* Π^1_2-CA_0. *Suppose that* X *is a metrizable countably based MF space. There is an embedding* f *of* X *as a dense subset of a complete separable metric space* $\widehat{A}$ *and a sequence* $\langle U_i \rangle$ *of open sets of* $\widehat{A}$ *such that* $f(X) = \bigcap_i U_i$.

The proof of this lemma is inspired by the proof of Theorem 8.17(ii) in [5]. That proof uses the strong Choquet game, which is not definable in second-order arithmetic. The proof presented here constructs the sequence of open sets directly.

Proof. Let d be a metric on X compatible with the original topology and let $A = \langle a_i \rangle$ be a dense subset of X. By Lemma 4.11, there is an isometric embedding f of X into $\langle \widehat{A}, d \rangle$.

We may use Π^1_2 comprehension to form the set $\{\langle a, r, p \rangle \mid B(a, r) \subseteq N_p\}$ and the set $\{\langle a, r, p \rangle \mid N_p \subseteq B(a, r)\}$. Here, $B(a, r)$ denotes a subset of X. We will use these sets as oracles for the rest of the proof.

We now construct a countable tree T consisting of certain finite sequences of the form $\langle W_0, q_0, W_1, q_1, \ldots, W_{n-1}, q_n \rangle$ or $\langle W_0, q_0, W_1, q_1, \ldots, q_n, W_n \rangle$ such that each W_i is a sequence of balls $B(a, r)$, with $a \in A$ and $r \in \mathbb{Q}^+$; $q_i \in P$ and $\operatorname{diam}(q_i) < 2^{-i}$ for $i \leq n$; and $N_{W_0} \supseteq N_{q_1} \supseteq N_{W_1} \supseteq N_{q_2} \supseteq \cdots$ and $q_0 \succeq q_1 \succeq q_2 \succeq \cdots$. We will ensure that for each n the set of all W which occur in the final position of a sequence of length $2n + 1$ in T forms a point-finite covering of X.

The tree is constructed inductively. At stage n, we put sequences of length $n + 1$ into the tree. We begin at stage 0 by forming a point-finite covering of X. For each W in this covering we put the sequence $\langle W \rangle$ into T.

At stage $2n + 1$, we first form the set S of pairs $\langle \sigma, B \rangle$ such that $\sigma = \langle W_0, q_1, W_1, \ldots, W_n, q_n \rangle$ is in T and $B \subseteq N_{q_n}$. We may make an enumeration $\{B_{\sigma,n}\}$ such that $\langle \sigma, B \rangle \in S$ if and only if there is an n such that $B = B_{\sigma,n}$.

Use the arithmetical functional given in Lemma 4.10 to form a point-finite refinement of the countable collection of open sets $\{B_{\sigma,n}\}$. The refinement replaces each ball $B_{\sigma,n}$ with an open set $W_{\sigma,n}$. Put into T every sequence of the form $\sigma \frown \langle W_{\sigma,n} \rangle$.

At stage $2n + 2$, for each sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n, W_{n+1} \rangle \in T$ put into T every sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n, W_{n+1}, q \rangle$ such that $q \preceq q_n$ and there exists a ball $B \in W_{n+1}$ such that $N_q \subseteq B$.

This completes the construction of T. We use T to construct a sequence $\langle U_n \mid n \in \mathbb{N} \rangle$ of open sets. For each n, U_n is the union of all open sets W which occur as the final open set in a sequence of length $2n + 1$ in T. Each

U_n may be written as the union of a set $\{B_i^n \mid i \in \mathbb{N}\}$ of basic open balls. A simple induction shows that for all $n \in \mathbb{N}$ the set U_n covers X; the key fact is that if $x \in B$ and $x \in N_p$ then there are p' with arbitrarily small diameter such that $x \in N_{p'} \subseteq B$ and $p' \preceq p$. For any basic open ball $B = B(a, r)$ in X we let $\widehat{B}$ denote the ball $B(a, r)$ in $\widehat{A}$; thus $f(x) \in \widehat{B} \Leftrightarrow x \in f(B)$ for each $x \in X$ and each basic open ball B. For each n we form an open subset of $\widehat{U}_n$ of $\widehat{A}$ by letting $\widehat{U}_n = \{\widehat{B}_i^n \mid i \in \mathbb{N}\}$.

We show that $f(X) = \bigcap_{n\in\mathbb{N}} \widehat{U}_n$ in $\widehat{A}$. Because each U_n covers X, each $\widehat{U}_n$ covers $f(X)$; this shows that $f(X) \subseteq \bigcap_{n\in\mathbb{N}} \widehat{U}_n$. Suppose that $\hat{z}$ is a Cauchy sequence in $\bigcap_{n\in\mathbb{N}} \widehat{U}_n$. Consider the set of all sequences in T of the form $\langle W_0, \ldots, W_{n+1}\rangle$ such that $\hat{z} \in \widehat{W_{n+1}}$. These sequences form a subtree $T_{\hat{z}}$ of T, because if $\hat{z} \in \widehat{W_{n+1}}$ then $\hat{z} \in \widehat{W_n}$ for any sequence $\langle W_0, \ldots, W_n, q_n, W_{n+1}\rangle \in T$. The tree $T_{\hat{z}}$ is infinite, because $\hat{z} \in U_n$ for all $n \in \mathbb{N}$. $T_{\hat{z}}$ is also finitely branching, because the collection of all W appearing at a level of the form $2n+1$ in T is point-finite. We apply König's Lemma to obtain a path $\langle W_0, q_o, W_1, q_1, \cdots\rangle$ through $T_{\hat{z}}$; it is clear that $\hat{z} \in \bigcap_i \widehat{W_i}$. Moreover, at most one point is in the intersection, because $\operatorname{diam}(q_i) < 2^{-i}$ for all $i \in \mathbb{N}$. The descending sequence $\langle q_i\rangle$ extends to a maximal filter $x \in X$, and this maximal filter is clearly in $\bigcap_i W_i$. Thus $f(x) = \hat{z}$. We have now shown $\bigcap_{n\in\mathbb{N}} \widehat{U}_n \subseteq f(X)$. □

We are now prepared to prove MFMT.

Theorem 4.2. *The following is provable in* Π^1_2-CA_0. *A countably based MF space is homeomorphic to a complete separable metric space if and only if it is regular.*

Proof. The classical proof that a complete metric space is regular goes through in Π^1_2-CA_0. Thus Π^1_2-CA_0 proves that if a countably based MF space is homeomorphic to a complete separable metric space then the MF space is regular.

The converse direction is proved as follows. Let X be a regular countably based MF space. By Theorem 4.1, X is metrizable. To see that X is completely metrizable, we apply Lemmas 4.12 and 4.9. □

4.2 Reversals of metrization theorems

We have shown above that Π^1_2-CA_0 proves that a countably based MF space is completely metrizable if and only if it is regular. In this section, we show in Π^1_1-CA_0 that Π^1_2 comprehension is required to prove this result. A sketch of the proof may be found in [14]. A complete proof appeared in the author's PhD thesis [13]. We give a complete proof here.

Theorem 4.3. *The proposition MFMT, which states that every countably based regular MF space is homeomorphic to a complete separable metric space, is equivalent to* Π^1_2-CA_0 *over* Π^1_1-CA_0.

Proof. The forward direction of the equivalence is given in Theorem 4.2. We work in Π^1_1-CA_0 and prove the reversal. The following technical lemma will be used to simplify our exposition; the lemma follows directly from the existence of a universal Π^1_2 formula and Kondo's Π^1_1 uniformization theorem, both of which are provable in Π^1_1-CA_0 (see Theorems VI.2.6 and VII.4.14 of [18]).

Lemma 4.13. *The following is provable in* Π^1_1-CA_0. *There is a* Π^1_1 *formula* $\Psi(n, f, h)$ *with one free number variable and two free set variables such that for each* $n \in \mathbb{N}$ *and* $h \in \mathbb{N}^\mathbb{N}$ *there is at most one* $f \in \mathbb{N}^\mathbb{N}$ *such that* $\Phi(n, f, h)$ *holds, and the existence of the set* $\{n \mid \exists f \Psi(n, f, h)\}$ *for every* h *implies* Π^1_2-CA_0.

We form a countable poset P as follows. Let Ψ be the Π^1_1 formula given by Lemma 4.13. Let $h \in \mathbb{N}^\mathbb{N}$ be fixed. For each n the set

$$C_n = \{f \in \mathbb{N}^\mathbb{N} \mid \Psi(n, f, h)\}$$

is coanalytic, so we may apply Theorem 2.8 to form a poset P_n. Note that each C_n contains at most one point, and is thus closed.

Because Theorem 2.8 is uniform, we may form the sequence $\langle P_n \mid n \in \mathbb{N}\rangle$ of these posets. Let P be the disjoint union of the posets $\langle P_n\rangle$, with an order such that $p \perp q$ if $p \in P_n$, $q \in P_m$, and $n \neq m$. Thus $\mathrm{MF}(P)$ is the topological disjoint union of the spaces $\langle \mathrm{MF}(P_n)\rangle$. In particular, the set $C = \bigcup_n C_n$ is closed in $\mathrm{MF}(P)$.

The properties of Ψ satisfy the hypothesis of Theorem 2.8 to ensure that P_n is regular for each n; thus P is regular. By assumption, there is a homeomorphism f from $\mathrm{MF}(P)$ to a complete separable metric space $\widehat{A}$.

A result of Brown [1] shows in Π^1_1-CA_0 that each closed subset of a complete separable metric space has a countable dense subset. Let D be a countable dense subset of C. Because each point of C is isolated in C, we have $D = C$. We may thus form the set

$$\{n \mid f(C_n) \text{ is nonempty}\} = \{n \mid C_n \text{ is nonempty}\} = \{n \mid \exists f \Psi(n, f, h)\}.$$

This completes the proof. □

A countably based MF space X is *completely metrizable* if there is a continuous function $d\colon X \times X \to [0, \infty)$ such that d is a complete metric on X compatible with the original topology.

Corollary 4.1. *The proposition that every countably based regular MF space is completely metrizable is equivalent to* Π^1_2-CA_0 *over* Π^1_1-CA_0.

Proof. The reversal is parallel to that in Theorem 4.3. It is only necessary to check that the hypothesis "X is homeomorphic to a complete separable metric space" may be weakened to "X is completely metrizable." □

Theorem 4.4. *The proposition that every countably based regular MF space is strongly regular is equivalent to* Π^1_2-CA_0 *over* Π^1_1-CA_0.

Proof. The following statement is provable in Π^1_1-CA_0.

> (∗) Every strongly regular metrizable countably based MF space is completely metrizable.

To show this, it is necessary to reprove the statement in Lemma 4.12 in Π^1_1-CA_0 with the added hypothesis of strong regularity. This is done in Lemma 4.3.29 of [13]. Once the modified version of Lemma 4.12 has been established, (∗) follows immediately. The reversal then follows from Corollary 4.1. □

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Many of the results in this paper appear in my PhD thesis [13]. A specific reversal (Theorem 4.3) has appeared in a paper by Mummert and Simpson [14].

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