On the existence of a connected component of a graph

Dagstuhl Seminar: Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis

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Introduction

We look at the Reverse Mathematics and Reverse Analysis of specific principles involving connected components of a countable graph.

This is joint work with Kirill Gura and Jeff Hirst.

Conventions

We consider only countably infinite graphs, and so we may always take the vertex set to be $\mathbb{N}$.

- A **countable graph** is represented by the adjacency relation between its vertices.
- A **connected component** is a maximal path connected subset of the graph. Formally, this is stated directly in terms of paths.
- An **antichain** is a set of vertices no two of which are in the same component.
- A **decomposition** of a graph G into connected components is a function $f\colon G \to G$ such that a and b are in the same component if and only if $f(a) = f(b)$.

Reverse Mathematics

Subsystems for Reverse Mathematics

Traditional Reverse Mathematics is carried out using particular subsystems of second order arithmetic.

- RCA_0 is the usual base system for Reverse Mathematics. It consists of PA^-, computable comprehension and Σ^0_1 induction.

- ACA_0 is an extension of RCA_0 adding the comprehension scheme for arithmetically definable sets (with parameters).

- $\mathsf{I}\text{-}\Sigma^0_2$ is the axiom scheme for induction on Σ^0_2 formulas (with parameters).

Existence of connected components

Theorem

The following are equivalent over RCA_0*.*

1. ACA_0*.*
2. *Every countable graph has a decomposition into connected components* [*Hirst 1992*]*.*
3. *Every countable graph has a connected component* [*GHM 2015*]*.*

Graphs with finitely many components

It is easier to build a connected component when the graph has only finitely many components.

Theorem

RCA_0 *proves that if a countable graph* G *has a finite set of vertices* V_0 *such that every vertex in* G *is path connected to some vertex in* V_0*, then* G *can be decomposed into components. In particular,* G *has a connected component.*

Graphs with finitely many components (2)

We can strengthen the principle by weakening the hypothesis.

Theorem

The following are equivalent over RCA_0.

1. $I\text{-}\Sigma^0_2$.
2. *If a countable graph* G *has a number* $n \in \mathbb{N}$ *such that among all sets of* n *vertices there are at least two that are connected by a path, then* G *can be decomposed into connected components.*
3. *Item (2) with "can be decomposed into connected components" replaced with "has a connected component".*

Constructing antichains

Theorem

The following are equivalent over RCA_0.

1. ACA_0
2. *If a countable graph has no infinite connected component, then it has an infinite antichain.*

In particular, there is a computable graph in which every component is finite that has no infinite c.e. antichain.

Weak Partitions

A **weak partition** is an sequence $P = (P_i : i \in \mathbb{N})$ of nonempty enumerated subsets of $\mathbb{N}$ such that, for all i and j, if $P_i \cap P_j \neq \emptyset$ then $P_i = P_j$.

A **component** of a weak partition (P_i) is a set X such that $X = P_k$ for some $k \in \mathbb{N}$.

Graph components and weak partitions

Theorem

The following are equivalent over RCA_0*.*

- *Every countable graph has a connected component.*
- *Every weak partition of* $\mathbb{N}$ *has a component.*

This theorem doesn't do justice the the equivalence.

Both principles are equivalent to ACA_0 over RCA_0,
so they are trivially equivalent to each other.

Reverse Analysis and Weihrauch Reducibility

Reverse Analysis / Weihrauch Reducibility

We can refine our equivalence theorems, emphasizing their uniformity, via Weihrauch reducibility.

A Π^1_2 problem P is **strongly Weihrauch reducible** to a Π^1_2 problem Q if there are computable functions Φ and Ψ such that $\Phi(f)$ is an instance of Q whenever f is an instance of P, and such that $\Psi(g)$ is a P-solution of f whenever g is a Q-solution of $\Phi(f)$.

Two Π^1_2 problems P, Q are **strongly Weihrauch equivalent**, written $P \equiv_{sW} Q$, if each is strongly Weihrauch reducible to the other.

Weihrauch reductions as translations

One goal of Reverse Mathematics is to study the underlying combinatorics of mathematical principles.

Proof-theoretic equivalence gives only a very coarse way of measuring this combinatorial similarity.

Strong Weihrauch reductions are somewhat analogous to bijective proofs from finite combinatorics.

They not only tell that two problems are generally equivalent, but show precisely how each instance of each problem can be viewed, up to uniform computable transformations, as an instance of the other.

Three problems

We consider the following Π^1_2 problems:

- D: decompose a countable graph into connected components.
- P: construct a connected component of a countable graph.
- WP: construct a component of a weak partition of $\mathbb{N}$.

Calibrating principles

We use a few "standard principles" to calibrate the strength of P, D, and WP.

- LPO: the "limited principle of omniscience": given $p\colon \mathbb{N} \to \mathbb{N}$, find an m such that $p(m) = 0$ if and only if $p(k) = 0$ for all k.
- $\widehat{LPO}$: given a sequence $(p_i : i \in \mathbb{N})$ of instances of LPO, find a corresponding sequence $(m_i : i \in \mathbb{N})$ of solutions.
- $C_{\mathbb{N}}$: "closed choice for $\mathbb{N}$": given a non-surjective map $p\colon \mathbb{N} \to \mathbb{N}$, find an element of $\mathbb{N}$ not in the range of p.

Strong Weihrauch Equivalences (1)

Theorem

The following equivalences hold:

$$P \equiv_{sW} Q \equiv_{sW} WP \equiv_{sW} \widehat{LPO}.$$

Theorem

The following equivalences hold:

$$P \equiv_{sW} \widehat{P}$$
$$Q \equiv_{sW} \widehat{Q}$$

Here $\widehat{P}$ *and* $\widehat{Q}$ *are infinite parallelizations of* P *and* Q*, defined in a way similar to* $\widehat{LPO}$*.*

Strong Weihrauch Equivalences (2)

Let P_k, D_k, and WP_k be the restrictions of P and D to graphs and weak partitions with exactly k components.

No additional information about the components is provided.

Theorem ($\star$)

The following equivalences hold:

$$P_k \equiv_{sW} D_k \equiv_{sW} WP_k \equiv_{sW} C_{\mathbb{N}}.$$

This theorem is of interest because it uses $C_{\mathbb{N}}$, a principle that is provably solvable in RCA_0, to measure the strength of a principle that has nontrival **induction** strength over RCA_0.

Question

One interest in Weihrauch-type reducibilities is to help measure the uniformity of proofs, as in the results here.

Another interest to find separations for principles that are ω-equivalent in Reverse Mathematics.

To what extent can we also use Weihrauch-type reducibilities to detect nontrivial *induction* strength in principles, as in $(\star)$ above?

To what extent are variants of $C_{\mathbb{N}}$ relevant to that project? For example, choice for co-finite subsets of $\mathbb{N}$ is related to a variant of the infinite pigeonhole principle.

Example 2

Consider this Π^1_2 problem:

- *WPHP*: a weak form of the infinite pigeonhole principle: if $(A_i : i \leq k)$ is a finite sequence of finite sets, find an upper bound for $\bigcup_{i \leq k} A_i$.

Theorem

$WPHP \equiv_{sW} C_{\mathbb{N}}$.

The possibility of solving $C_{\mathbb{N}}$ is verified in RCA_0.

The possibility of solving *WPHP* is not verifiable in RCA_0.
A theorem of Hirst shows that WPHP is equivalent to B-Σ^0_2.

Theorem

$WPHP \equiv_{sW} C_F$ ($\equiv_{sW} C_{\mathbb{N}}$ by Prop. 3.3 of [BG2011]).

Proof.

To see that $C_F \leq_{sW} WPHP$, let $A \subseteq \mathbb{N}$ be cofinite. Let $B = \mathbb{N} \setminus A$. We may view B as a sequence of one set, and use WPHP to find an upper bound n for B. Then $n + 1 \in A$, as desired.

To see that $WPHP \leq_{sW} C_F$, let $(A_i : i \leq k)$ be a finite sequence of finite sets. Let $U = \bigcup_{i \leq k} A_i$. Then $W = \mathbb{N} \setminus U$ is confinite, and every element of W is an upper bound for U, as desired.

□

References

Additional references are given in [GHM 2015].

[BG 2011] Vasco Brattka and Guideo Gherardi, "Effective choice and boundedness principles in computable analysis", Bulletin of Symbolic Logic 17, 2011, 73–117.

[Hirst 1992] Jeffry L. Hirst, "Connected components of graphs and Reverse Mathematics", Arch. Math. Logic 31, 1992, 183–192.

[GHM 2015] Kirill Gura, Jeffry L. Hirst and Carl Mummert, "On the existence of a connected component of a graph", Computability 4, 2015, 103–117.