The Pennsylvania State University

The Graduate School

Department of Mathematics

ON THE REVERSE MATHEMATICS OF GENERAL TOPOLOGY

A Thesis in

Mathematics

by

Carl Mummert

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2005

The thesis of Carl Mummert was reviewed and approved* by the following:

Stephen G. Simpson
Professor of Mathematics
Thesis Advisor
Chair of Committee

Dmitri Burago
Professor of Mathematics

John D. Clemens
Assistant Professor of Mathematics

Martin Fürer
Associate Professor of Computer Science

Alexander Nabutovsky
Professor of Mathematics

Nigel Higson
Professor of Mathematics
Chair of the Mathematics Department

*Signatures are on file in the Graduate School.

Abstract

This thesis presents a formalization of general topology in second-order arithmetic. Topological spaces are represented as spaces of filters on partially ordered sets. If P is a poset, let $\mathrm{MF}(P)$ be the set of maximal filters on P. Let $\mathrm{UF}(P)$ be the set of unbounded filters on P. If X is $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$, the topology on X has a basis $\{N_p \mid p \in P\}$, where $N_p = \{F \in X \mid p \in F\}$. Spaces of the form $\mathrm{MF}(P)$ are called MF spaces; spaces of the form $\mathrm{UF}(P)$ are called UF spaces. A poset space is either an MF space or a UF space; a poset space formed from a countable poset is said to be countably based. The class of countably based poset spaces includes all complete separable metric spaces and many nonmetrizable spaces including the Gandy–Harrington space. All poset spaces have the strong Choquet property.

This formalization is used to explore the Reverse Mathematics of general topology. The following results are obtained.

RCA_0 proves that countable products of countably based MF spaces are countably based MF spaces. The statement that every G_δ subspace of a countably based MF space is a countably based MF space is equivalent to Π^1_1-CA_0 over RCA_0.

The statement that every regular countably based MF space is metrizable is provable in Π^1_2-CA_0 and implies ACA_0 over RCA_0. The statement that every regular MF space is completely metrizable is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0. The corresponding statements for UF spaces are provable in Π^1_1-CA_0, and each implies ACA_0 over RCA_0.

The statement that every countably based Hausdorff UF space is either countable or has a perfect subset is equivalent to ATR_0 over ACA_0. Π^1_2-CA_0 proves that every countably based Hausdorff MF space has either countably many or continuum-many points; this statement implies ATR_0 over ACA_0. The statement that every closed subset of a countably based Hausdorff MF space is either countable or has a perfect subset is equivalent over Π^1_1-CA_0 to the statement that $\aleph_1^{L(A)}$ is countable for all $A \subseteq \mathbb{N}$.

Contents

Acknowledgements

I am indebted to Steve Simpson, my advisor, for his guidance and patience during my years as a graduate student. I am especially grateful for the significant amount of time that he has invested in me. His input as I have completed this thesis has been invaluable.

I would like to acknowledge the National Science Foundation, which provided financial support for two years of my graduate studies through a VIGRE grant to the Penn State Department of Mathematics.

Chapter 1

Introduction

1.1 Overview

This thesis is a contribution to the program of Reverse Mathematics, an ongoing research program in mathematics established in the 1970s by Harvey Friedman and Stephen Simpson. One goal of the program is to formalize large parts of core mathematics in second-order arithmetic. A second goal is to determine which set existence axioms are required to prove these theorems. In this setting, core mathematics means the mathematics learned by undergraduate mathematics majors, including calculus, real analysis, geometry, abstract algebra, combinatorics, basic mathematical logic, basic measure theory, and general topology.

Second-order arithmetic is a formal system in which, informally, there are natural numbers and sets of natural numbers but nothing else. Thus in second-order arithmetic there are no sets of sets or functions from sets to sets. Despite the seemingly limited supply of objects, second-order arithmetic is a rich and powerful system in which almost all the results of core mathematics can be established. A variety of (more or less natural) coding techniques are used to represent objects such as complete separable metric spaces as sets of natural numbers. Once these representations have been established, it is possible to state and prove theorems of core mathematics as theorems of second-order arithmetic.

A mathematical structure may be represented in second-order arithmetic if it is countable or has a countable substructure which determines the whole structure. Thus complete separable metric spaces, which are typically uncountable, may be represented because they are completely determined by a countable dense subset and a metric on the dense subset. This metric can be viewed as a sequence of real numbers, which in turn can be viewed as a single set of natural numbers. A detailed development of the number systems and complete separable metric spaces in a relatively weak subsystem of second-order arithmetic is given by Simpson [Sim99, Chapter II].

A central goal of Reverse Mathematics is to determine which set existence axioms are required to prove theorems of core mathematics. A set existence axiom is said to be required to prove a theorem if the theorem implies the axiom in a basic logical system strictly weaker than the axiom. A theorem is equivalent to a set existence axiom if the theorem requires the axiom and the axiom (with the possible addition of some weak basic axioms) is sufficient to prove the theorem. We may thus partition the theorems of core mathematics into equivalence classes of theorems that require the same set existence axioms for proof. A surprising fact is that a great many theorems of core mathematics fall into a few natural equivalence classes, as documented in [Sim99].

Core mathematics includes the basic definitions and results of general topology. These topics are part of the undergraduate curriculum at many universities and form part of the basic working knowledge of contemporary mathematicians. Despite this fact, there has been little previous research on formalizing general topology in second-order arithmetic. This thesis presents a specific formalization of general topology in second-order arithmetic. Topological spaces are coded as countable partially ordered sets (posets); the points in a space are represented by filters on the poset. Many familiar spaces can be represented in this way, including all complete separable metric spaces and many nonmetrizable spaces. This formalization of general topology is used to determine which set existence axioms are required to prove some basic theorems of general topology, including metrization theorems.

1.2 Previous and related research

The relationship between topological spaces and partially ordered sets has been known since Stone proved a famous duality theorem [Sto37]. Stone's theorem shows that the set of maximal filters on a Boolean algebra, with a natural topology determined by the algebra, forms a totally disconnected compact Hausdorff space. Furthermore, every such space is obtained from some Boolean algebra. Stone's work has inspired several contemporary research programs which use filters on partially ordered sets to represent topological spaces.

The program of *Locale Theory* (also known as pointfree topology or pointless topology) studies generalized objects known as locales using the tools of category theory; Johnstone [Joh82] provides a good introduction. Some authors, such as Fourman and Grayson [FG82], use an intuitionistic logic to study locales while other authors, such as Johnstone, use classical logic. A central goal of locale theory, according to Johnstone [Joh83], is to replace classical topological proofs with more constructive proofs in the category of locales.

The program of *Formal Topology* seeks to develop a theory of topology in intuitionistic type theory. A comprehensive introduction to this program is given by Sambin [Sam03]. Curi [Cur03] has shown that a version of Urysohn's Metrization Theorem for formal topologies has a constructive proof in type theory. We will show in Chapter 4 (see Corollary 4.3.3) that any proof of Urysohn's Metrization Theorem for MF spaces in second-order arithmetic requires nonconstructive set existence axioms.

An important distinction between the research in this thesis and the research in Locale Theory and Formal Topology is that we use classical logic and nonconstructive methods when they are necessary. We view our work as being formalized in set theory (or second-order arithmetic, which can be viewed naturally as a weak set theory). An even more crucial difference is that the research here is not meant to supplant the classical theory of topology or to correct deficiencies in the classical theory of topology. Instead, the research given here is intended to measure the set existence axioms required to prove well-known theorems of topology as they stand. Although we must formalize topology in second-order arithmetic to do this, our definitions of concepts such as continuous functions and metrizability are equivalent to the classical ones over ZFC.

The program known as *Domain Theory* uses classical logic to study various topologies on a class of partially ordered sets, known as *domains*, which have certain completeness properties. A thorough exposition of Domain Theory is given by Giertz *et al.* [GHK$^+$03]. The motivation behind Domain Theory is to consider maximal elements of a domain as "complete" or "total" objects, and view nonmaximal elements as approximations to complete objects. It has been shown that certain topological spaces may be represented as sets of maximal objects of domains. Lawson [Law97] has shown that every complete separable metric space may be represented as the set of maximal objects in a domain; Martin [Mar03] has shown that every metric space representable as the set of maximal objects of a domain is completely metrizable. There has been little work in the Domain Theory program to determine which set existence axioms are required to prove the theorems of

Domain Theory; a possible project in Reverse Mathematics is to formalize Domain Theory in second-order arithmetic and determine the strengths of theorems in Domain Theory. No such analysis is given in this thesis.

Contemporary research in the *Computable Analysis* program aims to develop a theory of computability for topological spaces. This research is inspired by the work of Bishop, Bridges, Weihrauch, and others. An important aspect of computable mathematics is that extra hypotheses are often required to prove effective versions of classical theorems. For example, it is common to assume that all continuous functions have a modulus of uniformly continuity. In this thesis, we formalize the theorems of mathematics without adding additional hypotheses. It is interesting to note that a result of Schröder [Sch98] inspired the proof of Urysohn's Metrization Theorem in second-order arithmetic presented in Chapter 4.

Many facts about complete separable metric spaces have been formalized in second-order arithmetic and analyzed for their their Reverse Mathematics strength. There has not been much work, however, on nonmetrizable spaces, or even on metrizable spaces without a fixed metric. Hirst [Hir93] has proved that the one-point compactification of a countable closed locally totally bounded subset of a complete separable metric space is metrizable. We obtain results in Section 4.4 which are closely related to Hirst's result.

1.3 Summary of results

In Chapter 2, we investigate a class of topological spaces known as poset spaces. This chapter is written in the style of contemporary core mathematics; it does not address issues of formalization. This chapter should thus be accessible to a broad audience of mathematicians.

There are two kinds of poset spaces: UF spaces and MF spaces. We show that the class of poset spaces includes all complete separable metric spaces and all regular locally compact Hausdorff spaces. Other spaces, including the Gandy–Harrington space, are also poset spaces. The class of MF spaces is closed under arbitrary topological products, and every G_δ subspace of an MF space is an MF space. Every metrizable poset space is completely metrizable. We show that every countably based Hausdorff MF space has either countably many points or else has cardinality $2^{\aleph_0}$. Countably based Hausdorff UF spaces either have countably many points or have a perfect subset.

In Chapter 3, we formalize poset spaces in second-order arithmetic. Formal definitions of coded poset spaces and coded continuous functions are presented and justified. In some cases, we must choose between definitions which are equivalent in ZFC but not equivalent in weak subsystems of second-order arithmetic. One example of such a choice is the definition of a metrizable poset space (Definition 3.2.25).

In Chapter 4, we explore the Reverse Mathematics of countably based poset spaces (that is, poset spaces obtained from countable posets). We first consider, in Section 4.1, the Reverse Mathematics of filter extension theorems. It is shown that ACA_0 is equivalent over RCA_0 to the statement that every subset of a countable poset has an upward closure. Π^1_1-CA_0 is equivalent over ACA_0 to the statement that every filter on a countable poset is contained in a maximal filter.

Section 4.2 gives a discussion of subspaces and product spaces of countably based MF spaces. It is shown that RCA_0 can construct products of countably based MF spaces. The statement that every G_δ subspace of a countably based MF space is a countably based MF space is equivalent to Π^1_1-CA_0 over RCA_0.

In Section 4.3, we examine the Reverse Mathematics of metrization theorems. We first examine Urysohn's Metrization Theorem. Urysohn's Metrization Theorem for MF spaces states that every countably based regular MF space is metrizable. This theorem is provable in Π^1_2-CA_0. Urysohn's Metrization Theorem for UF spaces is defined similarly. We show that Urysohn's Metrization Theorem for UF spaces is provable in Π^1_1-CA_0.

Choquet's Metrization Theorem states that a second-countable topological space is completely metrizable if and only if it is regular and has the strong Choquet property (see Definition 2.2.3). We prove in Section 2.3.3 that every poset space has the strong Choquet property. The strong Choquet property is not definable in second-order arithmetic. Our formalization of Choquet's Metrization Theorem for MF spaces is the statement "A countably based MF space is completely metrizable if and only if it is regular." Choquet's Metrization Theorem for UF spaces is defined similarly. We show in Section 4.3.2 that Π^1_2-CA_0 proves Choquet's Metrization Theorem for MF spaces, and Π^1_1-CA_0 proves Choquet's Metrization Theorem for UF spaces.

In Section 4.3.3, we show that Choquet's Metrization Theorem for MF spaces is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0. This is the first example of a theorem of core mathematics which is provable in second-order arithmetic and implies Π^1_2-CA_0. We also show that two other, closely related, theorems are equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

In Section 4.4, we explore compact poset spaces. We prove that ACA_0 is equivalent over WKL_0 to the statement that every sequence in a compact poset space has a convergent subsequence. We show in Π^1_1-CA_0 that every locally compact complete separable metric space has a one-point compactification.

In Section 4.5, we consider cardinality dichotomy theorems and perfect set theorems for poset spaces. We show that the perfect set theorem for countably based Hausdorff UF spaces is equivalent to ATR_0 over ACA_0. The perfect set theorem for analytic subsets of Hausdorff UF spaces is also equivalent to ATR_0 over ACA_0. A cardinality dichotomy theorem for countably based Hausdorff MF spaces is provable in Π^1_2-CA_0 and implies ATR_0 over ACA_0. We also show, in Π^1_1-CA_0, that the perfect set theorem for closed subsets of Hausdorff countably based MF spaces is equivalent to the proposition that $\aleph_1^{L(A)}$ is countable for all $A \subseteq \mathbb{N}$. This is an example of a natural statement about MF spaces which is independent of ZFC.

Chapter 2

Topology of Poset Spaces

In this chapter, we define and study two classes of topological spaces: MF spaces and UF spaces. The MF and UF spaces are collectively referred to as poset spaces. This chapter is written in the style of contemporary mathematics. Although the results presented here could be formalized in any sufficiently strong set theory, such as ZFC, we do not dwell on the formalization. In Chapters 3 and 4, we will formalize many of these results in the formal system of second-order arithmetic.

2.1 Background in general topology

The material in this section is standard and can be found in any introductory general topology book, such as those by Kelly [Kel55] or Hocking and Young [HY88]. These books give much more thorough expositions of topology than is given here; the purpose of this section is only to set the stage for the results to follow.

Definition 2.1.1. Let X be a nonempty set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X, known as the *open sets*, such that:

1. The empty set and the entire space are open sets.
2. If A is a collection of open sets then $\bigcup A$ is an open set.
3. If B is finite collection of open sets then $\bigcap B$ is an open set.

If a point x is in an open set U then we say U is a *neighborhood* of x. A set is called *closed* if its complement is open. The *closure* of a set S is the smallest closed set containing S; we denote it by $\text{cl}(S)$. A set is *clopen* if it is closed and open.

If $\mathcal{B} \subseteq \mathcal{T}$ and every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$ then $\mathcal{B}$ is called a *basis* for $\mathcal{T}$. A topological space is called *second countable* if it has a countable basis.

A *sequence* is a function from ω to X. Here $\omega = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. A sequence $\langle x_i \rangle$ is said to *converge* to a point $x \in X$ if x_i is eventually inside every open set containing x, i.e., if

$$\forall U \in \mathcal{T}\, [x \in U \Rightarrow \exists n \forall m > n (x_m \in U)].$$

In this case, we say that x is the *limit* of the sequence $\langle x_i \rangle$. The topological space X is called *separable* if there is a countable subset $A \subseteq X$ so that every point in X is the limit of a sequence of points in A.

Let X and Y be topological spaces. A function $f\colon X \to Y$ is called *continuous* if for every open set $V \in Y$ the preimage $f^{-1}(V)$ is open in X. It can be shown that a function $f\colon X \to Y$ is continuous if and only if whenever a point $x \in X$ satisfies $f(x) \in V$ for some open $V \subseteq Y$ there is an open neighborhood U of x such that $f(U) \subseteq V$. A continuous bijection with a continuous inverse is called a *homeomorphism*. If there is a homeomorphism from X to Y, we write $X \cong Y$. The homeomorphism relation is an equivalence relation on the class of topological spaces; the equivalence class of a space X under homeomorphism is called the *homeomorphism type* of X. Properties of a topological space which depend only on the homeomorphism type of the space are called *topological invariants*.

We now define what are known as the *separation axioms*. Let X be a topological space. We say X is a T_0 *space* if for every pair of distinct points at least one point has a neighborhood not containing the other. X is a T_1 *space* if for any two distinct points, each has a neighborhood not containing the other. X is a *Hausdorff space* if for any pair of distinct points x, y there is a pair of disjoint neighborhoods U, V with $x \in U$ and $y \in V$. X is a *regular space* if X is a T_0 space and whenever a point x is not in a closed set C there are disjoint open sets U, V with $x \in U$ and $C \subseteq V$. X is a *normal space* if X is a T_0 space and for every pair of disjoint closed sets C_0, C_1 there is a pair of disjoint open sets U_0, U_1 with $C_0 \subseteq U_0$ and $C_1 \subseteq U_1$. Note that the following implications hold for an arbitrary topological space:

$$\text{normal} \;\Rightarrow\; \text{regular} \;\Rightarrow\; \text{Hausdorff} \;\Rightarrow T_1 \Rightarrow T_0\,.$$

None of the implications in the other direction holds in general. Each of the separation axioms is a topological invariant.

A *metric* on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

For any $\epsilon > 0$ and $x \in X$, the *open ball* centered at x of radius ϵ (relative to d) is the set $B(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$.

Every metric on X induces a topology on X with the basis $\{B(x, \epsilon) \mid x \in X, \epsilon \in \mathbb{Q}^+\}$. A *metric space* $\langle X, d\rangle$ consists of a set X and a metric d on X. A topological space $\langle X, \mathcal{T}\rangle$ is *metrizable* if there is a metric d on X which induces $\mathcal{T}$. The following theorem, known as *Urysohn's Metrization Theorem*, gives a well known characterization of metrizability for second-countable spaces.

Theorem 2.1.2 (Urysohn). A second-countable space is metrizable if and only if it is regular.

Let $\langle X, d\rangle$ be a metric space. A sequence $\langle x_i\rangle \subseteq X$ is called a *Cauchy sequence* if

$$\forall \epsilon\, \exists N\, \forall n, m > N\, [d(x_n, x_m) < \epsilon].$$

A metric space is said to be *complete* if every Cauchy sequence converges to some point. Note that the completeness of a metric space depends on the specific metric given, not just on the homeomorphism type of the space. Thus completeness is not a topological invariant of metric spaces. The existence of a complete metric, however, is a topological invariant, which is known as *complete metrizability.* In Section 2.2.3, we will give a topological characterization (Theorem 2.2.5) of complete metric spaces, due to Choquet, which is analogous to Urysohn's Metrization Theorem for separable metric spaces.

A *perfect* subset of a topological space is a nonempty closed set which has no isolated points in the subspace topology. Every homeomorphic image of the Cantor space 2^ω is a perfect set, and every perfect set in a Hausdorff space contains a homeomorphic image of 2^ω. The cardinality of the Cantor space is $2^{\aleph_0}$ (the cardinality of the powerset of ω). Because a second-countable Hausdorff space has cardinality at most $2^{\aleph_0}$, the existence of a perfect set is enough to show that the cardinality of such a space is exactly $2^{\aleph_0}$.

A collection $\langle U_i \mid i \in I\rangle$ of open subsets of a topological space X is *point finite* if for each $x \in X$ the set $\{i \mid x \in U_i\}$ is finite. It is known that for each collection $\langle U_i\rangle$ of open subsets of a metric space there is a point-finite collection $\langle V_i\rangle$ such that $\bigcup_{i\in I} U_i = \bigcup_{i\in I} V_i$ and $V_i \subseteq U_i$ for each $i \in I$.

2.2 Games

In this section, we consider several *games* on topological spaces. These games are described in more detail in the monograph of Kechris [Kec95]. We will use these games to establish more complex topological properties of poset spaces in Section 2.3.3. A unique aspect of our analysis is that we will formalize the star game in second-order arithmetic in Chapter 4. The strong Choquet game cannot be directly formalized in second-order arithmetic, but this game is the inspiration for our proof of Lemma 4.3.23 in a subsystem of second-order arithmetic.

2.2.1 Gale–Stewart games

We begin with an informal description of Gale–Stewart games: these are games of length ω with perfect information. Each game is played in ω stages, numbered $0, 1, 2, \ldots$. There are two players: I and II. There is a set S of objects and a fixed $G \subseteq S^\omega$; each of these sets is known by both players before the game begins.

At stage n of the game, I chooses $a_n \in S$ and then II chooses $b_n \in S$. The elements of S are known as *moves*. Each players is aware of both moves that were made at each past stage, and each player may use a particular move an arbitrary number of times. The action of this game is often described in a diagram:

$$\begin{array}{llllll} \text{I:} & a_1 & & a_2 & & \cdots \\ \text{II:} & & b_1 & & b_2 & \cdots. \end{array}$$

At the end of the game, the players have chosen an infinite sequence

$$f = \langle a_1, b_1, a_2, b_2, \ldots \rangle \in S^\omega .$$

Each such sequence, known as a *play*, represents a single completed instance of the game. The set G, known as the *payoff set*, will tell which player won. If $f \in G$ then I wins. Otherwise, II wins. There are no ties.

For some games, one player will have a *winning strategy*. A winning strategy for I is a function $s_{\mathrm{I}} \colon \bigcup_{n\in\omega} S^{2n} \to S$ such that for every $f \in S^\omega$ if $f(2n+1) = s_{\mathrm{I}}(f[2n])$ for all n then $f \in G$. This means that I wins every play of the form

$$\begin{array}{llllll} \text{I:} & a_1 = s_{\mathrm{I}}(\langle\,\rangle) & & a_2 = s_{\mathrm{I}}(\langle a_1, b_1\rangle) & & a_3 = s_{\mathrm{I}}(\langle a_1, b_1, a_2, b_2\rangle) & \cdots \\ \text{II:} & & b_1 & & b_2 & & \cdots. \end{array}$$

A winning strategy for II is a function $s_{\text{II}}\colon \bigcup_{n\in\omega} S^{2n+1} \to S$ such that for every $f \in S^\omega$ if $f(2n+2) = s_{\text{II}}(f[2n+1])$ for all n then $f \notin G$. This means that II wins every play of the form

$$\begin{array}{llllll} \text{I:} & a_1 & & a_2 & & a_3 \quad \cdots \\ \text{II:} & & b_1 = s_{\text{II}}(\langle a_1\rangle) & & b_2 = s_{\text{II}}(\langle a_1, b_1, a_2\rangle) & \quad \cdots. \end{array}$$

At most one player can have a winning strategy; for some games neither player has a winning strategy. A game is *determined* if one player has a winning strategy. We say that the payoff set G is determined if its corresponding game is determined.

Theorem 2.2.1 (Gale–Stewart [GS53]). Let S be any set with the discrete topology and let G be a subset of S^ω. If G is either open or closed in the product topology on S^ω then G is determined.

2.2.2 Star games

Definition 2.2.2. We define the *star game* for an arbitrary topological space. Player I plays pairs of nonempty open sets $\langle V_i(0), V_i(1)\rangle$. Player II chooses an element of $\{0,1\}$ on each turn. So the game looks like:

$$\begin{array}{lllllll} \text{I:} & \langle V_0(0), V_0(1)\rangle & & \langle V_1(0), V_1(1)\rangle & & \langle V_2(0), V_2(1)\rangle & \cdots \\ \text{II:} & & k_0 & & k_1 & & k_2 \quad \cdots \end{array}$$

We require that

$$V_{i+1}(j) \subseteq V_i(k_i) \text{ for all } i \in \omega \text{ and } j = 0, 1. \tag{2.2.1}$$

The moves satisfying this requirement at any stage are called *legal* moves. Player I wins the game if requirement (2.2.1) is always met and $V_k(0) \cap V_k(1) = \varnothing$ for all k. The set of moves and the set of winning plays for I are thus well defined.

The star game on an arbitrary topological space is determined, because the set of winning plays for I is a closed subset of the set of all plays.

We informally describe the idea behind the star game. Player I is trying to construct an injective map from 2^ω to X; a winning strategy for I would give such an injection. The moves of player II localize the construction. Thus if II has a winning strategy then there is no open set in X where an injection can be constructed. It is a nonobvious fact (see Lemma 2.3.26) that for many spaces the existence of a winning strategy for II implies that the space is countable.

2.2.3 Strong Choquet games

A topological space has the *property of Baire* if every intersection of countably many dense open sets is dense. Every complete metric space has the property of Baire, and there are nonmetrizable spaces that have the property of Baire. There are metrizable spaces with the property of Baire that are not completely metrizable (see [Coh76] and [FK78]).

Definition 2.2.3. We define the *strong Choquet game* on a topological space X. Player I makes moves of the form $\langle U, x\rangle$ where U is an open set and $x \in U$. Player II plays open sets. So the game looks like this:

$$\begin{array}{lccccccc} \text{I:} & \langle U_0, x_0\rangle & & \langle U_1, x_1\rangle & & \langle U_2, x_2\rangle & & \cdots \\ \text{II:} & & V_0 & & V_i & & V_2 & \cdots \end{array}$$

We require that

$$V_i \subseteq U_i,\ U_{i+1} \subseteq V_i, \text{ and } x_i \in V_i \text{ for all } i \in \omega\,. \tag{2.2.2}$$

Thus $\bigcap U_i = \bigcap V_i$. Player I wins the game if $\bigcap U_i = \varnothing$; II wins otherwise. The set of moves and the set of winning plays for I are thus well defined.

A *strong Choquet space* is a space for which II has a winning strategy in the strong Choquet game.

The next theorem is well known.

Theorem 2.2.4. Every strong Choquet space has the property of Baire.

Proof. Assume that II has a winning strategy s for the strong Choquet game on X, and let $\langle U_i\rangle$ be a sequence of dense open sets. Let V be any open subset of X. At stage 0, choose choose a point $x_0 \in V \cap U_0$, and let W_0 be $s(\langle\langle x_o, V \cap U_0\rangle\rangle)$. By induction, assume that W_i has been defined. Choose a point $x_{i+1} \in W_I \cap U_{i+1}$ and let

$$W_{i+1} = s\left(\langle\langle x_0, V \cap U_0\rangle, W_0, \langle x_1, W_0 \cap U_1\rangle, W_1, \ldots, \langle x_i, W_i \cap U_{i+1}\rangle\rangle\right).$$

Because s is a winning strategy, there is a point in $x \in \bigcap W_i$. Clearly, $x \in V \cap \bigcap U_i$ as well. Thus $\bigcap U_i$ is dense. □

The motivation behind the strong Choquet game is clearest in the case where X is a metric space. Player I is trying to show that the metric space is not complete; so I will try to choose a Cauchy sequence $\langle x_i\rangle$ which does not converge. Player II tries to steer the developing sequence away from any "holes" in the space. If I has a winning strategy then II must be unable to prevent I from finding a hole, so the space must be "very" incomplete. If II has a winning strategy, then the holes are not "too dense." This informal explanation is justified by the following theorem of Choquet, which is proved in [Kec95, Theorem 8.17(ii)]. We will present a proof of a closely related statement in Lemma 4.3.23.

Theorem 2.2.5 (Choquet). A metric space is completely metrizable if and only if it is a strong Choquet space.

Unlike the star game, the strong Choquet game is not always determined. The complete metric spaces are a natural class of spaces for which II has a winning strategy. The Gandy–Harrington space is also a strong Choquet space. Player I has a winning strategy for the strong Choquet game on $\mathbb{Q}$.

We end the section by showing that, with regard to the existence of winning strategies, there is no loss in requiring players in the strong Choquet game to play open sets from a fixed basis.

Proposition 2.2.6. Consider the variant of the strong Choquet game in which both players are constrained to play open sets from a fixed basis B rather than playing arbitrary open sets. The variant game is determined if and only if the original game is determined, and in this case the same player has a winning strategy in both games.

Proof. First suppose I has a winning strategy for the original game. To play the variant game, I merely replaces the open set the strategy specifies with a basic open subset containing the specified point. Then II will respond with a basic open set containing the point and contained in the open set that the strategy wanted I to play. So I can use the strategy to get the next move, and so on.

Now suppose II has a winning strategy for the original game. Given an open set and a point by I, II chooses a basic open set containing the point and inside the open set prescribed by the strategy. Player I will respond with another open set and point inside the open set that II played. This move will be inside the open set the strategy prescribed. So II can continue using the original strategy to win.

We have thus shown that a winning strategy for the original game can be used by the same player to win the variant game. We prove the other direction of the implication to finish the proof.

Suppose that I has a winning strategy for the variant game. Then given an open set by player II, I merely replaces it with a basic open subset which would be a legal move for II, and applies the original strategy. It is straightforward to show that I will win the game.

Finally, suppose that II has a winning strategy in the variant game. Given an open set and a point by I, II replaces the open set with a basic open subset containing the point. This replacement yields a legal move for I; so II can use the strategy to choose the next move. This method will allow II to win the original game. □

2.3 Poset spaces

In this section we define the class of poset spaces. We show that all complete metric spaces are poset spaces, but not all second-countable Hausdorff poset spaces are metrizable. We also show that all locally compact Hausdorff spaces are poset spaces. In the remainder of the section, we establish some basic topological properties of poset spaces.

2.3.1 Definition and examples

Definition 2.3.1. A *poset* is a nonempty set P with a binary relation $\preceq$ such that for all $p, q, r \in P$:

1. $p \preceq p$.
2. If $p \preceq q$ and $q \preceq p$ then $q = p$.
3. If $p \preceq q$ and $q \preceq r$ then $p \preceq r$.

A poset P has an auxiliary relation $\prec$ defined by letting $p \prec q$ if $p \preceq q$ and $p \neq q$. The relations $\preceq$ and $\prec$ are so closely related that we will often use the most convenient relation for the task at hand without comment.

Each subset A of a poset P has an *upward closure*, denoted $\text{ucl}(A)$, satisfying $\text{ucl}(A) = \{p \in P \mid \exists q \in A[q \preceq p]\}$. If $A = \text{ucl}(A)$ then A is said to be *upward closed*.

Let P be a poset and let $p, q \in P$. If there is an $r \in P$ such that $r \preceq p$ and $r \preceq q$ then we say that p and q are *compatible*. If p and q are not compatible, they are said to be *incompatible*; in this case, we write $p \perp q$. If $p \preceq r$ then we say p *extends* r. Thus $p \perp q$ if and only if p and q have no common extension.

Definition 2.3.2. A *filter* is a subset $F \subseteq P$ such that for all $p, q \in P$:

1. If $p \in F$ and $q \in F$ then there is an $r \in F$ such that $r \preceq p$ and $r \preceq q$.

2. If $p \in F$ and $p \preceq q$ then $q \in F$.

In other words, a filter is an upward-closed subset of P in which every two elements have a common extension. We say that a filter F' *extends* a filter F if $F \subseteq F'$.

A filter is *maximal* if it cannot be extended to a strictly larger filter. We let $\mathrm{MF}(P)$ denote the set of all maximal filters on P.

A filter F on a poset P is *unbounded* if there is no $p \in P$ such that $p \prec q$ for all $q \in F$. We let $\mathrm{UF}(P)$ denote the set of all unbounded filters on P.

It is clear that every maximal filter is unbounded; thus $\mathrm{MF}(P) \subseteq \mathrm{UF}(P)$ for every poset P. The existence of maximal filters is a consequence of Zorn's Lemma.

A sequence $\langle p_i \mid i \in \omega \rangle$ is said to be *descending* if $p_{i+1} \preceq p_i$ for all $i \in \omega$. The sequence is *strictly descending* if $p_{i+1} \prec p_i$ for all $i \in \omega$.

Proposition 2.3.3. Let P be a countable poset and let F be a filter on P. There is a descending sequence $\langle p_i \mid i \in \omega \rangle \subseteq F$ such that for every $q \in F$ there is an $i \in \omega$ such that $p_i \preceq q$. Hence F is the upward closure of $\langle p_i \rangle$.

Definition 2.3.4 (Poset spaces). Let P be a poset. We define a topology on $\mathrm{UF}(P)$ by fixing as a basis the collection $\{N_p \mid p \in P\}$, where

$$N_p = \{F \in \mathrm{UF}(P) \mid p \in F\}.$$

We give $\mathrm{MF}(P)$ the topology inherited as as subspace of $\mathrm{UF}(P)$. Thus each $p \in P$ gives a neighborhood $N_p = \{F \in \mathrm{MF}(P) \mid p \in F\}$. Although the notation N_p is used both for $\mathrm{UF}(P)$ and for $\mathrm{MF}(P)$, context will always determine which kind of space is intended. For $U \subseteq P$, we let $N_U = \bigcup_{p \in U} N_p$.

A space of the form $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$, with the topology just described, is called a *poset space*. Spaces of the form $\mathrm{MF}(P)$ are called *MF spaces*, while spaces of the form $\mathrm{UF}(P)$ are called *UF spaces*. A poset space which is formed from a countable poset is said to be *countably based*.

The intuition motivating the definition of MF spaces is that each open set in a topological space determines a property that an arbitrary point in the space may or may not have (the property is that of membership in the open set). A collection of properties determined by a collection of open sets is consistent if and only if the open sets have a nonempty intersection. The T_1 axiom implies that the family of neighborhoods of a fixed point corresponds to a maximal consistent collection of properties. The definition of MF spaces reverses this reasoning. We begin with a poset P and declare that a collection $S \subseteq P$ is a consistent set of conditions if S extends to a filter on P. We construct a topological space from P by declaring that every maximal filter on the poset corresponds to exactly one point; we topologize the space of maximal filters by choosing the smallest (weakest) topology in which every element of the poset determines an open set. The T_1 axiom will always hold in the resulting space.

The intuition behind UF spaces is more subtle; there is no reason to believe that every unbounded filter of open sets in a topological space should determine a unique point. It is easy to construct a filter of open subsets of $\mathbb{R}$ (ordered by set inclusion) such that no open set is contained in every element of the filter, but nevertheless there are uncountably many points which lie in every open set of the filter. It is possible, however, to order the open sets of $\mathbb{R}$ in a different way so that the unbounded filters in the new ordering form a space homeomorphic to $\mathbb{R}$. More generally, every completely metrizable space can be represented as a UF space and as an MF space (see Theorem 2.3.9). Our motivation for studying UF spaces is that they have certain definability properties in second-order arithmetic that MF spaces do not have.

Lemma 2.3.5. Every UF space satisfies the T_0 axiom. Every MF space satisfies the T_1 axiom. Moreover, for any poset P, $\mathrm{UF}(P)$ satisfies the T_1 axiom if and only if $\mathrm{UF}(P) = \mathrm{MF}(P)$.

Proof. Because any two distinct filters on P are distinct as subsets of P, their symmetric difference is nonempty. Thus there is an element of P which belongs to one filter but not the other. This shows that $\mathrm{MF}(P)$ and $\mathrm{UF}(P)$ are T_0 spaces for every poset P.

For any distinct maximal filters F and G on P there must be $p, q \in P$ such that $p \in F \setminus G$ and $q \in G \setminus F$. This shows that each MF space is a T_1 space.

If $\mathrm{UF}(P) = \mathrm{MF}(P)$ then $\mathrm{UF}(P)$ is a T_1 space. To prove the converse, suppose that $\mathrm{UF}(P)$ satisfies the T_1 axiom. Let F be an unbounded filter. Towards a contradiction, assume there is a filter G strictly extending F; then G is also unbounded. By the T_1 axiom, there is a $p \in F \setminus G$, which is a contradiction. Thus $\mathrm{UF}(P) = \mathrm{MF}(P)$ if and only if $\mathrm{UF}(P)$ is a T_1 topological space. □

The previous lemma is optimal in the sense that an arbitrary UF space need not be T_1, and an arbitrary MF space need not be Hausdorff. We now give examples to prove this claim.

Examples 2.3.6. We construct a countable poset P such that $\mathrm{UF}(P)$ is not a T_1 space. Let $\{p_i\}$ and $\{q_i\}$ be disjoint sets. We let $\prec$ be the smallest partial order on P such that:

1. $p_i \prec p_j$ whenever $i > j$.
2. $q_i \prec q_j$ whenever $i > j$.
3. $q_i \prec p_i$ for all i.

The unbounded filter $\{p_i\}$ has no open neighborhood not containing the unbounded filter $\{q_i\}$. Thus $\mathrm{UF}(P)$ is not T_1, and $\mathrm{UF}(P) \neq \mathrm{MF}(P)$.

We next construct a countable poset Q such that $\mathrm{MF}(Q)$ is not a Hausdorff space. Let $\{p_i\}$, $\{q_i\}$, and $\{r_i\}$ be pairwise-disjoint countable sets and let $P = \{p_i\} \cup \{q_i\} \cup \{r_i\}$. The order $\prec$ on Q is the smallest partial order on Q such that:

1. $p_i \prec p_j$ whenever $i > j$.
2. $q_i \prec q_j$ whenever $i > j$.
3. $r_i \perp r_j$ whenever $i \neq j$.
4. $r_i \prec p_j$ whenever $i > j$.
5. $r_i \prec q_j$ whenever $i > j$.

It is straightforward to verify that $F_1 = \{p_i\}$ and $F_2 = \{q_j\}$ are distinct maximal filters on P. But there are no incompatible p, q such that $F_1 \in N_p$ and $F_2 \in N_q$. Thus $\mathrm{MF}(Q)$ is not a Hausdorff space.

Example 2.3.7. We construct a countable poset P such that $\mathrm{MF}(P)$ is Hausdorff but not regular. Let $A = \{a_\sigma \mid \sigma \in 2^{<\omega}\}$ and $B = \{b_\tau \mid \tau \in 2^{<\omega}\}$ be disjoint sets. We write $|\sigma|$ for the length of $\sigma \in 2^{<\omega}$.

The elements of the poset P are exactly the disjoint union of A and B. The order on P is the smallest order containing the following relations:

1. $a_\sigma \preceq a_{\sigma'}$ whenever $\sigma \supseteq \sigma'$.
2. $b_\tau \preceq b_{\tau'}$ whenever $\tau \supseteq \tau'$.
3. $b_\tau \preceq a_\sigma$ if $\tau = \sigma$ and the last element in the sequence σ is a 1.

We begin by characterizing the maximal filters on P. Let f be an element of 2^ω. For each $n \in \omega$ let $f[n] = \langle f(0), \ldots, f(n-1) \rangle \in 2^{<\omega}$. If f is eventually zero, then the set $\{a_{f[n]} \mid n \in \mathbb{N}\}$ is a maximal filter on P. Otherwise, $\{a_{f[n]}, b_{f[n]} \mid n \in \omega\}$ is a maximal filter on P. These are all of the maximal filters on P.

We next show that $\mathrm{MF}(P)$ is Hausdorff. Suppose that F and G are distinct maximal filters. If both F and G contain an element of B, then F and G contain incompatible elements of B, and these yield a pair of disjoint neighborhoods of F and G. If neither F nor G contains an element of B, a similar argument shows that F and G have disjoint neighborhoods. Now suppose G contains an element of B and F does not. Choose n such that $f(m) = 0$ for all $m > n$. Choose $k > n$ such that $g(k) = 1$. Then $a_{f[k+1]}$ and $b_{g[k+1]}$ yield disjoint neighborhoods of F and G, respectively.

Finally, we show that $\mathrm{MF}(P)$ is not regular. Let g be the unique element of 2^ω such that $g(n) = 1$ for all $n \in \omega$. Let G be the associated filter. Note that $N_{b_{\langle\rangle}}$ is a neighborhood of G. We claim that there is no $p \in P$ such that $G \in N_p$ and $\mathrm{cl}(N_p) \subseteq N_{b_{\langle\rangle}}$. If there were such a neighborhood, there would be a neighborhood of the form N_{b_τ} for some $\tau \in 2^{<\omega}$. Let $\tau \in 2^{<\omega}$ be fixed. Let f be the unique element of 2^ω such that $f[|\tau|] = \tau$ and $f(n) = 0$ for all $n \geq |\tau|$, and let F be the associated filter. Clearly $F \notin N_{b_\tau}$. We claim that $F \in \mathrm{cl}(N_{b_\tau})$. Let a_σ be an arbitrary neighborhood of F; we may assume $|\sigma| > |\tau|$ and thus $\tau \subseteq \sigma$. Let g_σ be the unique element of 2^ω such that $g_\sigma[|\sigma|] = \sigma$ and $g_\sigma(n) = 1$ for $n \geq |\sigma|$. Then $g_\sigma \in N_{a_\sigma}$ and, since $\tau \subseteq \sigma$, $g_\sigma \in N_{b_\tau}$. Thus N_{a_σ} and N_{b_τ} are not disjoint. Since no neighborhood of F is disjoint from N_{b_τ}, $F \in \mathrm{cl}(N_{b_\tau})$.

It is interesting to note that the subset of $\mathrm{MF}(P)$ corresponding to $Z = \{f \in 2^\omega \mid \exists n \, \forall m \geq n \, [f(m) = 0]\}$ is closed in $\mathrm{MF}(P)$, but Z is an F_σ set in 2^ω. The construction of P is closely related to a more general construction which will be presented as Lemma 4.3.35. We will show in Example 4.3.38 how to obtain the present example as a corollary of Lemma 4.3.35.

We do not know if the space constructed in Example 2.3.7 is homeomorphic to a UF space; see Problem 2.3.34.

Open Problem 2.3.8. Find an example of a countable poset P such that $\mathrm{UF}(P)$ is Hausdorff but not regular.

We will next show that the class of poset spaces is quite rich by proving that several large classes of topological spaces may be represented as poset spaces. The first such class consists of the completely metrizable spaces. We let $\mathbb{Q}^+$ denote the set of positive rational numbers.

Theorem 2.3.9. Suppose that X is a completely metrizable space. There is a poset P such that $X \cong \mathrm{MF}(P)$ and $\mathrm{UF}(P) = \mathrm{MF}(P)$. We may take the cardinality of P to be that of any dense subset of X.

Proof. Let d be a complete metric on X compatible with the original topology. We may assume that X is infinite, because the theorem is trivial if X is finite. Let A be a dense subset of X. Let $P = A \times \mathbb{Q}^+$, and order P by setting $\langle a, r\rangle \prec \langle b, s\rangle$ if $d(a, b) + r < s$. Note that if $\langle a, r\rangle \prec \langle b, s\rangle$ then $\mathrm{cl}(B(a, r)) \subseteq B(b, s)$, although the converse may fail (in a discrete space, for example). It is clear that the cardinality of P is that of A, since A is infinite.

We define a map $\phi\colon X \mapsto \mathrm{MF}(P)$ which will turn out to be a homeomorphism. For each $x \in X$, choose a sequence $\langle x_i\rangle$ of elements of A such that $x_i \to x$ and $d(x_i, x) \leq 2^{-i}$ for all i; this is possible even if X is not separable, because X is a metric space. Let $\phi(x)$ be the filter generated by $\{\langle x_i, 2^{-i}\rangle \mid i \in \omega\}$; denote this filter $F(\langle x_i\rangle)$.

We first prove that $\phi(x)$ is well defined. Suppose that $\langle x_i\rangle$ and $\langle y_j\rangle$ are two sequences satisfying the conditions of the previous paragraph. Let N_p be in $F(\langle x_i\rangle)$; so $\langle x_i, 2^{-i}\rangle \preceq p$ for some i. Choose k such that $2^{-i} - d(x_i, x) > 2^{-k}$. Then

$$\begin{aligned} d(y_{k+1}, x_i) &\leq d(y_{k+1}, x) + d(x_i, x) \\ &\leq 2^{-(k+1)} + d(x_i, x), \end{aligned}$$

which implies

$$\begin{aligned} d(y_{k+1}, x_i) + 2^{-(k+1)} &\leq 2^{-k} + d(x_i, x) \\ &< 2^{-i}. \end{aligned}$$

This implies that $\langle y_{k+1}, 2^{-k+1}\rangle \preceq \langle x_i, 2^{-i}\rangle$; hence $\langle x_i, 2^{-i}\rangle \in F(\langle y_j\rangle)$. We conclude $F(\langle x_i\rangle) \subseteq F(\langle y_j\rangle)$. A parallel argument shows that $F(\langle y_j\rangle) \subseteq F(\langle x_i\rangle)$. Hence ϕ is well defined.

Next, we show that $\phi(x)$ is a maximal filter. Suppose that $\{\langle y, r\rangle\} \cup \phi(x)$ extends to a filter F. Let $\langle\langle y_j, r_j\rangle \mid i \in \mathbb{N}\rangle$ be an infinite descending sequence in F beginning with $\langle y, r\rangle$ which is eventually below every element of $\phi(x)$. It is clear that the sequence $\langle y_j \mid i \in \mathbb{N}\rangle$ converges to x. It follows from the argument in the previous paragraph that $\langle y, r\rangle \in \phi(x)$. Thus $\phi(x)$ is a maximal filter.

We claim that $\phi\colon X \to \mathrm{MF}(P)$, which we have now shown to be well defined, is a homeomorphism. It is straightforward to show that ϕ is injective.

Given $F \in \mathrm{MF}(P)$, we can use any infinite descending sequence of elements of F to find a single point $x \in X$ such that $x \in \bigcap F$. But then $F = \phi(x)$. Thus ϕ is surjective. The preimage of a condition $\langle x, 2^{-i} \rangle \in P$ consists of those points $y \in X$ for which $d(x, y) < 2^{-i}$; hence $\phi^{-1}(N_{\langle x, 2^i \rangle}) = B(x, 2^{-i})$. Thus ϕ is continuous. The proof that ϕ is an open map is similar.

To see that $\mathrm{UF}(P) = \mathrm{MF}(P)$, note that in any unbounded descending sequence $F = \{\langle a_i, r_i \rangle\}$ we have $\lim r_i = 0$. Thus there is a unique $x \in \bigcap B(a_i, r_i)$; so $\mathrm{ucl}(F)$ is the maximal filter $F(x)$. □

Corollary 2.3.10. For every complete separable metric space X there is a countable poset P such that $X \cong \mathrm{MF}(P)$ and $\mathrm{MF}(P) = \mathrm{UF}(P)$.

A space is *locally compact* if every point has a open neighborhood with compact closure. It is known that every open subset of a compact Hausdorff space is locally compact. Conversely, for each locally compact Hausdorff space X there is a compact Hausdorff space known as the *one-point compactification* of X, and X is homeomorphic to an open subset of its one-point compactification. To obtain the one-point compactification of a locally compact Hausdorff space X, we add a single point x_∞ to X and place x_∞ in each open subset of X with compact complement. It is known that the one-point compactification of a locally compact complete separable metric space is a compact metric space.

Every locally compact second-countable space is completely metrizable, but there are separable compact Hausdorff nonmetrizable spaces (e.g., a space 2^X with X uncountable of cardinality $\leq 2^{\aleph_0}$) as well as non-locally-compact complete separable metric spaces (e.g., the Baire space ω^ω). Therefore, the next theorem does not imply, and is not implied by, Theorem 2.3.9. Note that a locally compact T_0 space is Hausdorff if and only if it is regular.

Theorem 2.3.11. Every locally compact Hausdorff space is homeomorphic to an MF space.

Proof. Let P consist of the open subsets of X with compact closure. For $p, q \in P$ we let $p \preceq q$ if $p = q$ or $\mathrm{cl}(p) \subseteq q$. Note that, because $p \subseteq \mathrm{cl}(p)$, if $p \preceq q$ and $q \preceq p$ then $p = q$. Therefore $\preceq$ is partial order relation on P.

For each point $x \in X$, we let

$$F(x) = \{p \in P \mid x \in p\}.$$

Claim 1. For any point $x \in X$, the set $F(x)$ is a filter. Given $p, q \in F(x)$, let r be the open set $p \cap q$. So $x \in r$. Choose an open neighborhood s of x such that $\mathrm{cl}(s) \subseteq r$ (using regularity). Then $\mathrm{cl}(s)$ is compact, because $\mathrm{cl}(p)$

is compact and $s \subseteq p$; also $\mathrm{cl}(s) \subseteq (p \cap q)$. Thus $s \in P$; $s \preceq p$ and $s \preceq q$; and $x \in s$. This shows that $F(x)$ is a directed set. It is clear that $F(x)$ is upward closed. This proves Claim 1.

Claim 2. For each point x the set $F(x)$ is a maximal filter. Given x, let p be any open set not in $F(x)$. Let y be any point in p. Use axiom T_1 to obtain open sets q and r such that $x \in q \setminus r$ and $y \in r \setminus q$. By intersecting, we may assume $r \subseteq p$.

Assume for a contradiction that we can add p to $F(x)$ and extend the result to a filter G. Since G contains q and r, and $r \not\preceq q$, G must contain an open set s with $s \preceq r$. This means $\mathrm{cl}(s) \subseteq r$. Since $x \notin r$, we see that $x \notin \mathrm{cl}(s)$. Choose an open set q' such that $x \in q'$ and $q' \cap s = \varnothing$. Now $q' \in G$ and $s \in G$, but $q' \perp s$. We have reached a contradiction, which completes the proof of Claim 2.

We have now proved that for each point x there is a unique maximal filter $F(x)$. We next show that each maximal filter of P is obtained as $F(x)$ for some x.

Claim 3. For each maximal filter F on P there is a point $x \in X$ such that $F = F(x)$. Given any filter F, let $F' = \{cl(s) \mid s \in F\}$. Any finite intersection of elements of F' is nonempty, because any finite collection of elements of F has a common extension in F. Therefore there is a point $x \in \bigcap F'$, because F' has the finite intersection property (recall that each $t \in F'$ is compact). We will show $x \in \bigcap F$.

First assume there is a minimal element $p \in F$. Then p must be a one-point open set, by an application of the T_1 and T_3 axioms. Now the T_1 axiom implies that every one-point set is closed. Therefore p is a closed set, so $x \in \mathrm{cl}(p) = p = \bigcap F$.

Now assume that there is no minimal element in F. Let $p \in F$ be fixed. Let $q \in F$ be an extension of p. Then $x \in \mathrm{cl}(q) \subseteq p$, so $x \in p$. Because p was arbitrary, we have shown $x \in \cap F$. Therefore, by maximality, $F = F(x)$.

We now show that the map $F \colon x \mapsto F(x)$ is a homeomorphism from X to $\mathrm{MF}(P)$. We have already shown the map is bijective. Recall that the open sets in P form a basis for X. For $p \in P$, we have (by duality and definition)

$$x \in p \quad \Leftrightarrow \quad p \in F(x) \quad \Leftrightarrow \quad F(x) \in N_p.$$

This implies that the map F is a homeomorphism. □

In Section 4.4, we will prove in second-order arithmetic that the one-point compactification of a locally compact complete separable metric space is a compact metric space.

If P is a countable poset then $\mathrm{UF}(P)$ and $\mathrm{MF}(P)$ are second-countable topological spaces. Establishing a converse to this statement is, apparently, difficult.

Open Problem 2.3.12. Let P be an uncountable poset such that $\mathrm{MF}(P)$ is a second-countable topological space. Must there be a countable poset R such that $\mathrm{MF}(P)$ is homeomorphic to $\mathrm{MF}(R)$? The corresponding question for UF spaces is also open.

2.3.2 Subspaces and product spaces

In this section, we show that the class of MF spaces is closed under arbitrary topological products. We also show that every nonempty G_δ subset of an MF space is itself an MF space.

Definition 2.3.13. Let $\langle P_i \mid i \in I \rangle$ be an indexed collection of posets. We define the *restricted product poset* $P = \widetilde{\prod}_{i\in I} P_i$. The elements of this poset are functions p such that the domain of p is a finite subset of I and $p(i) \in P_i$ for all i in the domain (note that $\widetilde{\prod} P_i$ is not a direct product). The order relation $\preceq$ on P is obtained by setting $p \preceq p'$ if and only if the domain of p includes the domain of p' and $p(i) \preceq_{P_i} p'(i)$ for all i in the domain of p'.

For each $i \in I$ we define a partial function $\pi_i : P \to P_i$ by letting $\pi_i(p) = p(i)$ if i is in the domain of p; $\pi_i(p)$ is undefined otherwise. For $S \subseteq \widetilde{\prod}_{i\in I} P_i$ and $i \in I$, we define

$$\pi_i(S) = \{q \in P_i \mid \text{ there is a } p \text{ in } S \text{ such that } \pi_i(p) = q\}.$$

We omit the straightforward proof of the following lemma.

Lemma 2.3.14. If $\langle P_i \mid i \in I \rangle$ is an indexed collection of posets and F is a filter on $\widetilde{\prod}_{i\in I} P_i$ then $\pi_i(F)$ is a filter on P_i for each $i \in I$.

We use the lemma to prove the class of MF spaces is closed under topological products.

Theorem 2.3.15. For any indexed collection $\langle P_i \mid i \in I \rangle$ of posets, the topological product space $\prod_{i\in I} \mathrm{MF}(P_i)$ is homeomorphic to $\mathrm{MF}(\widetilde{\prod}_{i\in I} P_i)$.

Proof. Let $F_i \in \mathrm{MF}(P_i)$ for $i \in I$. We define a set $F = \widetilde{\prod}_{i\in I} F_i$ as the collection of all finite sequences $\langle p_{i_1}, \ldots, p_{i_N} \rangle$ such that $p_{i_n} \in F_{i_n}$ and $i_j \neq i_k$ whenever $j \neq k$. It is straightforward to verify that F is a filter on $P = \widetilde{\prod}_{i\in I} P_I$, because F_i is a filter for each i. Moreover, $\pi_i(F) = F_i$ for all i.

Now suppose that F is not a maximal filter. Choose $p = \langle p_{i_1}, \ldots, p_{i_N} \rangle \in P \setminus F$ such that $F \cup \{p\}$ extends to a filter F'. There must be an $n \leq N$ such that $p_{i_n} \notin F_{i_n}$; otherwise $p \in F$. Let $F'_i = \pi_i(F')$. Then $F_i \subsetneq F'_i$, contradicting the assumption that F_i is maximal. This shows that F is a maximal filter. Hence the map $\phi\colon \prod_{i \in I} \mathrm{MF}(P_i) \to \mathrm{MF}(\widetilde{\prod}_{i \in I} P_i)$ defined by $\phi(\langle F_i \rangle) = \widetilde{\prod} F_i$ is well defined. It is straightforward to check that ϕ is a homeomorphism. □

Corollary 2.3.16. If $\langle P_i \mid i \in \omega \rangle$ is a sequence of countable posets then the topological product space $\prod_{i \in \omega} \mathrm{MF}(P_i)$ is homeomorphic to a space $\mathrm{MF}(P)$, where P is a countable poset.

We note that the product poset construction seems to require the use of maximal filters. It is not known if the class of UF spaces is even closed under finite topological products.

Open Problem 2.3.17. Suppose that P and Q are posets. Must there be a poset R such that $\mathrm{UF}(P) \times \mathrm{UF}(Q) \cong \mathrm{UF}(R)$? If P and Q are countable, can we take R to be countable?

A G_δ subspace of a topological space is the intersection of countably many open sets; an F_σ subspace is the union of countably many closed sets.

Theorem 2.3.18. For every poset P and every nonempty G_δ subspace U of $\mathrm{MF}(P)$ there is a poset Q such that $U \cong \mathrm{MF}(Q)$. If P is countable then we may take Q to be countable.

Proof. We may assume that P has no minimal elements. Let $\langle U_i \mid i \in \omega \rangle$ be a sequence of open sets such that $U = \bigcap_i U_i$ is a nonempty G_δ subspace of $\mathrm{MF}(P)$.

We define Q to be the set of pairs $\langle n, p \rangle \in \omega \times P$ such that $n > 0$, $p \in P$, $N_p \subseteq \bigcap_{j=0}^{n} U_j$, and $N_p \cap U \neq \varnothing$. We let $\langle n, p \rangle \prec_Q \langle m, q \rangle$ if $m < n$ and $p \prec q$. Clearly, Q is countable if P is countable.

Given $\langle n, p \rangle \in Q$, we may pick a point $x \in N_p \cap U$. Because $x \in U_{n+1}$, there is some q such that $x \in N_q \subseteq U_{n+1}$. Thus, if we let r be a common extension of p and q in x then $\langle n+1, r \rangle \in Q$ and $\langle n+1, r \rangle \prec_Q \langle n, p \rangle$. This shows that Q has no minimal elements.

Suppose that we are given a point $x \in U$. We define $F(x) \subseteq Q$ by

$$F(x) = \{\langle n, p \rangle \mid x \in N_p \text{ and } N_p \subseteq \textstyle\bigcap_{i=0}^{n} U_i\}\,.$$

This is the set of all conditions in Q which are compatible with x. Note that $F(x)$ is a filter.

We now show that $F(x)$ is a maximal filter. Suppose that there is some $\langle n, p\rangle \in Q$ such that $F(x) \cup \{\langle n, p\rangle\}$ extends to a filter G. Let $G' \subseteq P$ be the set of elements of P that appear in G. Clearly $x \subseteq G'$; since x is maximal, this means $x = G'$. Thus $x \in N_p$. We conclude $F(x)$ is maximal.

Claim: Let G be any maximal filter on Q. Let G' be the set of elements of P that appear in the conditions in G. Then the upward closure of G' is a maximal filter on P and the point represented by G' is in U.

It is straightforward to show that the upward closure of G' is a filter on P. Since there are no minimal elements in Q, the numbers n appearing in the conditions in G must be arbitrarily large; hence any point in the G_δ set coded by G' must be in U.

To finish our proof of the claim, we need to show that the upward closure of G' is a maximal filter. Suppose that $p \in P$ and $G' \cup \{p\}$ extends to a maximal filter x on P. This implies $G \subseteq F(x)$. Because G is maximal, $G = F(x)$. Thus $p \in G$. This completes the proof of the claim.

It only remains to check that F is a homeomorphism; this follows from the fact that $x \in p \Leftrightarrow \langle n, p\rangle \in F(x)$ whenever $(n, p) \in Q$. □

Theorem 2.3.18 is optimal in a certain sense: the set of rationals is not a poset space, but is an F_σ subset of the metrizable poset space $\mathbb{R}$. Note that closed subsets of nonmetrizable poset spaces need not be G_δ subsets; thus the theorem does not imply that every closed subset of an MF space is an MF space (see Section 4.5).

We will show in Corollary 2.3.21 that open subspaces of UF spaces are UF spaces. We do not know if G_δ subspaces of UF spaces are always poset spaces.

Open Problem 2.3.19. Let P be a poset and let U be an G_δ subset of $\mathrm{UF}(P)$. Must there be a poset Q such that $U \cong \mathrm{UF}(Q)$? Must there be a poset R such that $U \cong \mathrm{MF}(R)$? If P is countable, may we assume that Q or R is countable?

Theorem 2.3.18 implies that open subsets of MF spaces are themselves MF spaces. The next theorem gives a sharper result.

Theorem 2.3.20. Let U be a nonempty open subset of $\mathrm{MF}(P)$. There is a subposet Q of P such that U is homeomorphic to $\mathrm{MF}(Q)$.

Proof. Define $Q = \{q \in P \mid N_q \subseteq U\}$, and give Q the order induced as a subposet of P. We claim that the map $f\colon U \to \mathrm{MF}(Q)$ given by $x \mapsto x \cap Q$ is a homeomorphism. We first show that f is well defined. Suppose that $x \in U$, and assume that there is an $r \in Q$ such that $f(x) \cup \{r\}$ extends to a filter y on Q. Then $\mathrm{ucl}(y)$ is a filter on P which extends x and contains r; so $r \in x$. Thus $f(U) \subseteq \mathrm{MF}(Q)$.

It is clear that f is an injective, and that f is a continuous open map. We must show that f is surjective. Let $y \in \mathrm{MF}(Q)$, and let x be the upward closure of y in P. We claim that $x \in \mathrm{MF}(P)$. Let $r \in P$ be such that $x \cup \{r\}$ extends to a filter x' on P. Choose any $p \in y$ and let q be a common extension of p and r in x'. Then $q \in Q$, which means that $q \in Q \cap x'$. Thus $q \in y$, because y is a maximal filter. This shows that $r \in x$. We conclude x is maximal. □

Corollary 2.3.21. Let U be a nonempty open subset of $\mathrm{UF}(P)$. There is a subposet Q of P such that U is homeomorphic to $\mathrm{UF}(Q)$.

Proof. The proof is a straightforward modification of the proof of Theorem 2.3.20. □

2.3.3 Poset spaces and games

Definition 2.3.22. Let P be a poset. We define a variant of the star game (Definition 2.2.2) called the *poset star game*. To play the poset star game on P, both players play elements of P rather than open sets, we replace requirement (2.2.1) with the requirement

$$V_{i+1}(j) \preceq_P V_i(k_i) \text{ for all } i \in \omega \text{ and } j \in \{0, 1\},$$

and we let I win if this requirement is always followed and moreover $V_k(0) \perp V_k(1)$ for all $k \in \omega$.

Lemma 2.3.23. The poset star game on any poset is determined.

Proof. The legal moves for each player at any fixed stage form a closed subset of all the moves (because the topology on the set of moves is discrete). Hence the set of all plays with only legal moves is closed. The set of all plays such that $V_k(0) \perp V_k(1)$ for all k is closed. Therefore the set of winning plays for I is closed. By Theorem 2.2.1, the game is determined. □

We will now use the poset star game to obtain results on the cardinality of Hausdorff poset spaces. We will show that a Hausdorff countably based MF space is either countable or has cardinality $2^{\aleph_0}$, while a Hausdorff UF space is either countable or has a perfect subset.

Lemma 2.3.24. Let X be a Hausdorff poset space based on a countable poset P. Suppose that I has a winning strategy for the poset star game on P. Then X has cardinality $2^{\aleph_0}$.

Proof. It suffices to prove the result for $\mathrm{MF}(P)$, which is a subset of $\mathrm{UF}(P)$. Let s_{I} be a winning strategy for I and let $f \in 2^\omega$. Consider the play in which I follows s_{I} while II uses f as a guide; that is, II plays $f(n)$ at stage $2n$. Because s_{I} is a winning strategy for I, this play determines a descending sequence $F(f)$ of elements of P. This sequence extends to a maximal filter. For distinct $f, g \in 2^\omega$ the sequences $F(f)$ and $F(g)$ contain incompatible elements and thus cannot extend to the same filter. Therefore the space $\mathrm{MF}(P)$ has cardinality $2^{\aleph_0}$. □

Lemma 2.3.25. Let P be a countable poset in which I has a winning strategy for the poset star game. Assume $\mathrm{UF}(P)$ is Hausdorff. Then $\mathrm{UF}(P)$ has a perfect subset.

Proof. We show how the proof of Lemma 2.3.24 gives a perfect subset of $\mathrm{UF}(P)$. Recall that for each $f \in 2^\omega$ the strategy for I gives a filter on P; let $F(f)$ denote this filter. If f and g are distinct elements of 2^ω then the associated filters $F(f)$ and $F(g)$ contain incompatible elements.

Not every filter of the form $F(f)$ can be bounded. There are only countably many elements in P to serve as lower bounds, and no single element can be the lower bound for $F(f)$ and $F(g)$ if $f \neq g$. Let $B \subseteq 2^\omega$ consist of those f such that $F(f)$ is bounded. Then B is countable, so $2^\omega \setminus B$ contains a perfect subset D. It is straightforward to check that the set $\{F(g) \mid g \in D\}$ is a perfect subset of $\mathrm{UF}(P)$. □

Lemma 2.3.26. Let X be a countably based Hausdorff poset space based on the poset P. If II has a winning strategy for the poset star game on P then X is countable.

Proof. Let s_{II} be a winning strategy for II. We say that a finite play σ of length $2k$ is *compatible with* s_{II} if $s_{\mathrm{II}}(\sigma[2i+1]) = \sigma(2i+2)$ whenever $2i+2 \leq k$. We say that a play σ of even length is a *good play* for a point x if σ is compatible with s_{II} and x is in the open set chosen by II in the last move of σ. A good play for x is a *maximal play* if it cannot be extended to a longer good play for x; this means that no matter what pair of disjoint open sets I plays, s_{II} will direct II to choose an open set not containing x.

If II has a winning strategy then every point x has a maximal play. Note that the empty play is trivially a good play for x. If every good play for x

could be extended to a larger good play for x, then it would be possible for player I to win the game by always leaving the game in a position that is good for x. This play of the game would follow s_{II}, a winning strategy for II, which is a contradiction.

If σ is a good play for two points x and y then σ is not a maximal play for both x and y. For I could play $\langle U_1, U_2 \rangle$ in response to σ, where $x \in U_1$, $y \in U_2$, and $U_1 \cap U_2 = \varnothing$. Here we are using the assumption that the topology of X is Hausdorff.

We have now shown that every point in the X has a maximal play, and that no play is maximal for two points. Since the set of maximal plays is countable, this implies that the set of points in X is countable. □

Theorem 2.3.27. A countably based Hausdorff poset space either countable or has cardinality $2^{\aleph_0}$. If P is countable and $\mathrm{UF}(P)$ is an uncountable Hausdorff space then $\mathrm{UF}(P)$ contains a perfect subset.

Proof. By Lemmas 2.3.23–2.3.26. □

In Section 4.5, we determine which set existence axioms are required to prove a formalized version of Theorem 2.3.27 for Hausdorff UF spaces in second-order arithmetic.

Open Problem 2.3.28. Suppose that X is a countably based Hausdorff MF space which is uncountable. We have shown that X has cardinality $2^{\aleph_0}$. Must X have a perfect subset? We will show in Section 4.5 that the related proposition "Every closed subset of a countably based Hausdorff MF space is either countable or contains a perfect subset" is independent of ZFC.

We now use the strong Choquet game to show that every metrizable poset space is completely metrizable. Keye Martin [Mar03] has given a parallel analysis of topological spaces represented as maximal elements of a domain.

Theorem 2.3.29. Every poset space is a strong Choquet space.

Proof. We describe the strategy for II informally. At the start of the game, player I plays an open set U_0 and a point x_0. Player II translates the point x_0 into a filter on P, then finds a basic neighborhood q_0 of x such that $N_{q_0} \subseteq U_0$. Player II then plays N_{q_0}. Now given $\langle x_1, U_1 \rangle$ with $x_1 \in N_{q_0}$, II translates x_1 to a filter on P and then finds a neighborhood q_1 of x_1 such that $q_1 \preceq_P q_0$ and $N_{q_1} \subseteq U_1$. Player II plays N_{q_1}. Player II continues this strategy, always choosing $q_{i+1} \preceq_P q_i$. At the end of the game, II has determined $\{q_i \mid i \in \omega\}$, a descending sequence of elements of P. This sequence of elements extends to a maximal filter, giving a point $x \in \bigcap N_{q_i}$. Player II has thus won the game. $\square$

Corollary 2.3.30. Every poset space which is metrizable is completely metrizable.

Proof. Combine Theorems 2.3.29 and 2.2.5. $\square$

Corollary 2.3.31. Every poset space has the property of Baire.

Proof. Combine Theorems 2.3.29 and 2.2.4. $\square$

We now obtain a complete characterization of the metrizability and complete metrizability of countably based poset spaces.

Theorem 2.3.32. Let X be a countably based poset space. The following are pairwise equivalent:

1. X is a regular topological space.
2. X is metrizable.
3. X is completely metrizable.

Proof. The implication (1)⇒(2) follows from Urysohn's Metrization Theorem. The implication (2)⇒(3) follows from Theorem 2.2.5 and Theorem 2.3.29. We sketch a proof of the the final implication, (3)⇒(1), which is well known. Let d be a metric on X and let $B(x, r)$ be any open ball around $x \in X$, where $r \in \mathbb{Q}^+$; then $\text{cl}(B(x, r/2)) \subseteq B(x, r)$. It follows that X is regular. $\square$

There are countably based Hausdorff MF spaces which are not regular; such spaces cannot be metrizable. Thus statement (1) in Theorem 2.3.32 cannot be replaced with the statement that X is Hausdorff. One example of a nonregular Hausdorff countably based MF space was given in Example 2.3.7. Another example is the Gandy–Harrington space (see Section 2.3.4).

Theorems 2.3.15 and 2.3.18 have natural analogues for the class of complete separable metric spaces and for the class of strong Choquet spaces. It is well known that every G_δ subset of a complete separable metric is completely metrizable (we will prove this in RCA_0 in Lemma 4.3.18), and it is not difficult to show that a G_δ subset of a strong Choquet space is strong Choquet. Both the class of complete separable metric spaces and the class of strong Choquet spaces are closed under countable products. Every poset space is a strong Choquet space. These similarities between strong Choquet spaces and MF spaces suggest the following question.

Open Problem 2.3.33. Is every strong Choquet space homeomorphic to a poset space? Is every second-countable strong Choquet space homeomorphic to a countably based poset space?

Open Problem 2.3.34. Is there an MF space that is not homeomorphic to a UF space? If P is countable, must there be a countable poset Q such that $\mathrm{MF}(P) \cong \mathrm{UF}(Q)$? It follows from Theorem 2.3.32 and Corollary 2.3.10 that every metrizable MF space is a UF space. It is known that, under the set-theoretic assumption $\mathsf{V} = \mathsf{L}$, there is a countably based MF space that is not homeomorphic to any UF space.

Remark 2.3.35. Some well-known subspaces of product topologies are not strong Choquet spaces, and thus are not poset spaces.

Let Y be an infinite-dimensional Banach space. We let Y^*_w denote the dual space of Y, which consists of the bounded linear functionals on Y, with the weak-$*$ topology, that is, the topology of pointwise convergence. We now show that Y^*_w is not a strong Choquet space, and thus is not a poset space. Let $\langle c_i \mid i \in \mathbb{N}\rangle$ be an infinite linearly independent sequence of unit vectors in Y. Player I begins by playing the open set $U_1 = \{\phi \in Y^*_w \mid \phi(c_1) > 1\}$ and an element of $y_1 \in Y^*_w$ in this set; such a functional exists by the Hahn-Banach theorem. Player II plays an open subset of Y^*_w; I chooses a basic open subset V_1 of the set played by II such that $y_1 \in V_1 \subseteq U_1$. There will be an N_1 such that the basic open set does not restrict the values of the functional on y_n for $n \geq N_1$. Player I next plays the open set $U_2 = V_1 \cap \{\phi \in Y^*_w \mid \phi(c_{N_1}) > 2\}$ and a functional in it. In this way, I chooses a sequence of open sets $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ such that no functional in U_i has norm less than i. There can be no bounded linear functional in $\bigcap U_i$.

A similar proof shows that the space $C([0,1])$ of continuous real-valued functions on the unit interval with the topology of pointwise convergence is not a strong Choquet space.

2.3.4 The Gandy–Harrington space

Recall that $\omega = \{0, 1, 2, \ldots\}$ is the set of natural numbers. The standard topology on the space ω^ω is obtained by giving ω the discrete topology and giving ω^ω the product topology. There is another important topology on ω^ω which has more open sets than the product topology. The Gandy–

Harrington space is the set ω^ω with this alternate topology. This space is of great importance in descriptive set theory; see [Kec95], [MK80] and [HKL90, pp. 907–908]. The purpose of this section is to show that the Gandy–Harrington space is (homeomorphic to) an MF space. This result is particularly interesting to us because the Gandy–Harrington space is a nonmetrizable second-countable Hausdorff space.

Definition 2.3.36. A *computable tree on* $\omega \times \omega$ is a nonempty computable set of finite sequences of pairs of integers which is closed under taking initial segments. The word computable is used in the sense of computability theory, as described by Rogers [Rog67].

For T a computable tree on $\omega \times \omega$, we let $[T]$ denote the set of all paths through T. A function $f : \omega \to \omega \times \omega$ is a *path* through T if $\langle f(0), \dots, f(n)\rangle \in T$ for all $n \in \omega$. We define a projection $p_1 \colon [T] \to \omega^\omega$ by letting $(p_1(f))(n)$ be the first coordinate of the pair $f(n)$ for all $f \in [T]$ and $n \in \omega$. Let $p_1(T)$ denote the set of all projections of elements of $[T]$: $p_1(T) = \{p_1(f) \mid f \in [T]\} \subseteq \omega^\omega$.

We are now prepared to define the Σ^1_1 subsets of ω^ω, which will be the basic open sets of the Gandy–Harrington space. The definition given here is not the standard one in mathematical logic, but is equivalent to the standard definition (see [Sim99, Theorem V.1.7]).

Definition 2.3.37 (Σ^1_1 sets). A subset A of ω^ω is called Σ^1_1 if and only if there is a computable tree T on $\omega \times \omega$ such that $A = p_1(T)$.

The *Gandy–Harrington space* is the set ω^ω with the unique topology having as a basis the collection of Σ^1_1 subsets of ω^ω. This is a finer topology than the usual (product) topology on ω^ω.

Theorem 2.3.38. The Gandy–Harrington space is (homeomorphic to) a countably based MF space.

Proof. We begin by defining a poset P. We will show that the Gandy–Harrington space is homeomorphic to $\mathrm{MF}(P)$. For the purposes of this proof, a *condition* is a finite list of the form $\langle \sigma, \langle T_1, \tau_1\rangle, \langle T_2, \tau_2\rangle, \dots, \langle T_k, \tau_k\rangle\rangle$, where $\sigma, \tau_1, \tau_2, \dots, \tau_k \in \omega^{<\omega}$, $|\sigma| = |\tau_1| = \cdots = |\tau_k|$, and each T_i is a computable tree on $\omega \times \omega$. A condition is called *consistent with* $f \in \omega^\omega$ if $\sigma \subseteq f$ and there are functions $g_1, \dots, g_k \in \omega^\omega$ such that $\tau_i \subseteq g_i$ and $\langle f, g_i\rangle \in [T_i]$ for $i \leq k$. A condition is *consistent* if it is consistent with some f. Let P be the set of all consistent conditions.

Each consistent condition codes a nonempty Σ^1_1 set. Namely, the condition $\langle \sigma, \langle T_1, \tau_1 \rangle, \langle T_2, \tau_2 \rangle, \ldots, \langle T_k, \tau_k \rangle \rangle$ codes the set of all $f \in \omega^\omega$ such that $\sigma \subseteq f$ and for all $i \leq k$ there is a $g_i \in \omega^\omega$ such that $\tau_i \subseteq g_i$ and $\langle f, g_i \rangle \in [T_i]$.

Let $c = \big\langle \sigma, \langle T_1, \tau_1 \rangle, \ldots, \langle T_k, \tau_k \rangle \big\rangle$ and let $c' = \big\langle \sigma', \langle T'_1, \tau'_1 \rangle, \ldots, \langle T'_j, \tau'_j \rangle \big\rangle$. We define a strict partial order relation $\prec$ on P by declaring $c' \prec c$ if and only if $\sigma \subseteq \sigma'$, $|\sigma| < |\sigma'|$ and for each $i \leq k$ there is a $i' \leq j$ such that $T_i = T'_{i'}$ and $\tau_i \subseteq \tau'_{i'}$. It is straightforward to verify that $\prec$ is a partial order on P. Note that $c' \preceq c$ implies that the Σ^1_1 set coded by c' is a subset of the Σ^1_1 set coded by c, but the reverse implication is false. Also note that for every $c \in P$ there is a $c' \in P$ with $c' \prec c$.

We now define a homeomorphism ϕ between the Gandy–Harrington topology on ω^ω and $\mathrm{MF}(P)$. Note that each maximal filter F on P must contain an infinite descending sequence. For otherwise the filter would contain a $\prec$-minimal consistent condition, which is impossible. Therefore the elements σ appearing in F determine a unique f in ω^ω. Let $\phi(F) = f$.

Claim: If F is a filter on P containing the infinite descending sequence c_n, which determines a function f, and c is any condition in F then c is consistent with f. Without loss of generality we may assume that c is of the form $\big\langle \sigma, \langle T, \tau \rangle \big\rangle$. The claim is proved by using common extensions of c_n and c to build sequences $\langle \sigma_i \rangle$ and $\langle \tau_i \rangle$ such that in the end $f = \bigcup \sigma_i$ and $\bigcup \tau_i$ is a witness that $f \in p_1(T)$.

Fix $f \in \omega^\omega$ and let F be the set of all conditions consistent with f. The claim above shows that F is a maximal filter. Moreover $\phi(F) = f$; hence ϕ is surjective.

Now fix $F, G \in \mathrm{MF}(P)$ such that $\phi(F) = \phi(G)$. Choose any condition $c \in F$. By the claim, c is consistent with $\phi(F)$. Hence c is also consistent with $\phi(G)$. It follows that $G \cup \{c\}$ extends to a filter. Since G is maximal, this implies that $c \in G$. We have shown $G \subseteq F$; the reverse inclusion follows from the same argument. Therefore $F = G$. This shows that ϕ is injective.

For $c \in P$, the set $\phi(N_c)$ consists of exactly those $f \in \omega^\omega$ which are in the Σ^1_1 set coded by c. Hence the image of an open set under ϕ is open.

Let A be a Σ^1_1 subset of ω^ω. Let T_A be a computable tree on $\omega \times \omega$ such that $A = p_1(T_A)$. Then $\phi^{-1}(A) = N_c$ where $c = \big\langle \langle\,\rangle, \langle T_A, \langle\,\rangle \rangle \big\rangle$ (where $\langle\,\rangle$ denotes the empty sequence). Therefore ϕ is continuous. □

The following corollary, which is well known, follows from the previous theorem and Theorem 2.3.29.

Corollary 2.3.39. The Gandy–Harrington space has the strong Choquet property.

The next two theorems are well known. We sketch their proofs briefly.

Theorem 2.3.40. The Gandy–Harrington space is not regular, and thus not metrizable.

Proof sketch. Because the Gandy–Harrington space is second countable, it would be metrizable if it were were regular. If the space were metrizable, every closed subset of the topology would be a G_δ subset. It is well known that there are Π^1_1 (hence, closed in the Gandy–Harrington topology) subsets of ω^ω which are not boldface $\boldsymbol{\Sigma}^1_1$. Every G_δ subset of the Gandy–Harrington space is boldface $\boldsymbol{\Sigma}^1_1$, because every open set in the Gandy–Harrington space is boldface $\boldsymbol{\Sigma}^1_1$ and the collection of boldface $\boldsymbol{\Sigma}^1_1$ sets is closed under countable intersections. This contradiction shows that the Gandy–Harrington space cannot be regular. □

Theorem 2.3.41. Let $U = \{f \in \omega^\omega \mid \omega_1^f = \omega_1^{\mathrm{CK}}\}$. Then U is a dense open subset of the Gandy–Harrington space, and is completely metrizable in the subspace topology.

Proof sketch. One proof of this theorem is mentioned by Kechris and Louveau [KL97, p. 225]; we will sketch this proof briefly. An alternate proof, which does not use the strong Choquet property, is given by Hjorth [Hjo02, Section 2]. Hjorth's proof exhibits a homeomorphism between U and a G_δ subset of 2^ω, which allows a reasonably explicit complete metric on U to be defined.

It is well known that U is a Σ^1_1 set; thus U is an open subset of the Gandy–Harrington space. The fact that U is dense is the restatement of a well-known basis theorem for Σ^1_1 sets; a proof is given in Sacks [Sac90, Corollary 1.5]. The basis theorem may also be obtained as corollary of a basis theorem due to Gandy, Kriesel, and Tait [GKT60] (compare [Sim99, Corollary VII.2.12]).

To see that the Gandy–Harrington topology on U is regular, note that the intersection of U with any Π^1_1 set is a countable union of Δ^1_1 sets, and thus every Σ^1_1 subset of U is clopen. This shows that U has a clopen basis; any space with a clopen basis is regular. Since the Gandy–Harrington topology on U is also second countable, this topology is metrizable. As an open subset of a strong Choquet space, U inherits the strong Choquet property. Thus Choquet's theorem shows that U is completely metrizable. □

2.4 Remarks

In this chapter, we have developed the properties of poset spaces in ZFC set theory. Our true motivation for studying poset spaces is to formalize them in second-order arithmetic; it would not be interesting to formalize an overly limited class of spaces. We have shown that the class of countably based MF spaces includes all the complete separable metric spaces as well as many other spaces. This class of MF spaces is closed under taking countable products and G_δ subspaces, just as the class of complete separable metric spaces and the class of strong Choquet spaces are. These results show that we are justified as choosing poset spaces as our formalization of general topology in second-order arithmetic.

We will use this formalization to determine the Reverse Mathematics strength of metrization theorems. Urysohn's Metrization Theorem shows that every countably based regular poset space is metrizable. We have shown, as a consequence of Choquet's Metrization Theorem, that every metrizable countably based poset space is completely metrizable, and every separable complete metric space may be represented as a countably based MF space which is also a UF space; thus every metrizable countably based poset space will be representable in second-order arithmetic. This shows that the class of metrizable countably based poset spaces is the same in ZFC set theory as it is in second-order arithmetic.

Remark 2.4.1 (Filters and convergence). There are two ways of defining convergence in a topological space. One approach is via sequences (or, more generally, nets) of points; the other approach is via filters. In this context, a filter F is a collection of open sets such that the empty set is not in F, the intersection of two sets in F is in F, and any open superset of a set in F is in F. A filter F converges to a point if every open neighborhood of the point is in the filter. This terminology, established by Cartan [Car37a, Car37b], is well known.

Let us temporarily call the filters defined in the previous paragraph C-filters. We will use the term P-filters to refer to filters on posets, as we have defined them in Definition 2.3.2. Suppose that X is a poset space based on a poset P. We wish to compare the P-filters on P with the C-filters on X. To facilitate the comparison, we identify each $p \in P$ with the open set $N_p \subseteq X$. Note the the intersection of any two sets in a C-filter must be in the C-filter, while a P-filter satisfies the weaker condition that it contains an open subset of the intersection of any two open sets it contains. Each P-filter F thus generates the C-filter $\{U \mid U \text{ is open and } \exists p \in F\,[N_p \subseteq U]\}$.

Not every C-filter arises in this way, however; it can be seen that a C-filter is obtained from a P-filter if and only if there is a point of X in the intersection of the open sets in the C-filter. This illustrates the role of the partial ordering on P: it must control descending sequences of poset elements so that the intersection of the corresponding open sets remains nonempty. This difference is highlighted by the classical theorem that a topological space is compact if and only if every maximal C-filter on the space converges. By definition, every maximal P-filter converges, even if $\mathrm{MF}(P)$ is not compact.

Chapter 3

Formalized Poset Spaces

The previous chapter considered poset spaces from the point of view of contemporary mathematics. In the present chapter, we shift our point of view to that of mathematical logic. We formalize poset spaces in second-order arithmetic. We assume the reader has some familiarity with the essentials of mathematical logic; one good reference is by Monk [Mon76].

3.1 Subsystems of second-order arithmetic

This section presents a survey of subsystems of second-order arithmetic. Because the results in this section are well known, we present them without proof. The monograph by Simpson [Sim99] gives a complete reference for subsystems of second-order arithmetic and includes proofs of all the results stated here.

Despite its name, second-order arithmetic is formalized in first-order logic. The language has two kinds of variables. Variables of the first kind are called *number variables* and are intended to range over $\omega = \{0, 1, 2, 3, \ldots\}$. Variables of the second kind are called *set variables* and are intended to range over $P(\omega)$, the collection of all subsets of ω. We use uppercase Roman letters to denote set variables and lowercase Roman letters to denote number variables, with the exception that x, y, and z always denote set variables.

Definition 3.1.1. The *language of second-order arithmetic*, denoted L_2, consists of:

1. Binary function symbols $+$ and $\cdot$ for numbers.

2. Constant number symbols 0 and 1.

3. A membership relation, $\in$, which is intended to tell if a number is in a set.

4. A binary equality relation, $=$, for number variables.

5. A binary order relation, $<$, for number variables.

Equality for sets is not included in L_2, but is defined by extensionality:

$$A = B \equiv \forall n\, [n \in A \Leftrightarrow n \in B].$$

Definition 3.1.2. We will denote the formal system of second-order arithmetic by Z_2. The axioms of Z_2 fall into three groups. The first group consists of the *basic axioms* of first-order number theory, without induction. These axioms are the universal closures of the following formulas:

1. $n + 1 \neq 0$
2. $m + 1 = n + 1 \Rightarrow m = n$
3. $m + 0 = m$
4. $m + (n + 1) = (m + n) + 1$
5. $m \cdot 0 = 0$
6. $m \cdot (n + 1) = m \cdot n + m$
7. $\neg(m < 0)$
8. $m < n + 1 \Leftrightarrow (m < n \vee m = n)$

The second group of axioms is the *comprehension scheme*, which consists of the universal closure of

$$\exists S\, \forall n\, [n \in S \Leftrightarrow \Phi(n)]$$

for each formula $\Phi(n)$ of L_2 in which S is not free. The third group of axioms is the *induction scheme*, which consists of the universal closure of

$$[\Phi(0) \wedge \forall n(\Phi(n) \Rightarrow \Phi(n + 1))] \Rightarrow \forall n\, \Phi(n)$$

for each L_2 formula Φ. This axiomatization of second-order arithmetic differs from the one given in [Sim99] but is equivalent to the one given there.

Remark 3.1.3. We will always assume without comment the axioms of classical logic, including the axiom of the excluded middle and the substitution properties of the equality relation. For example, the sentence

$$\forall n \, \forall m \, \forall X \, [(n \in X \wedge m = n) \Rightarrow m \in X]$$

is an axiom of classical logic.

Definition 3.1.4 (Bounded quantifiers). The *numeric terms* are the smallest class of expressions in the language L_2 containing the constant number symbols $0, 1$ and the number variables such that $t_1 + t_2$ and $t_1 \cdot t_2$ are numeric terms whenever t_1 and t_2 are numeric terms.

Let Φ be any L_2 formula. Let t be any numeric term in which the variable n does not appear. We introduce the *bounded quantifiers* $\forall n < t$ and $\exists n < t$. By definition, $\forall n < t \, \Phi$ is an abbreviation for the formula $\forall n[n < t \Rightarrow \Phi]$, while $\exists n < t \, \Phi$ abbreviates the formula $\exists n[n < t \wedge \phi]$.

Definition 3.1.5 (Classification of L_2 formulas). There is a classification of L_2 formulas based on the type and number of alternating quantifiers. Let Φ be any L_2 formula.

- Φ is called *arithmetical* if Φ has no set quantifiers.
- Φ is called Δ^0_0 if Φ has no unbounded quantifiers.
- Φ is called Σ^0_k, for $k \in \omega$, if Φ is equivalent to a formula of the form

  $$\exists n_1 \, \forall n_2 \, \exists n_3 \, \cdots \, Q_k n_k \, \Theta,$$

  where there are k alternating number quantifiers beginning with $\exists$ and Θ is Δ^0_0. Here Q_k is $\forall$ if k is even, $\exists$ if k is odd.
- Φ is called Π^0_k, for $k \in \omega$, if Φ is equivalent to a formula of the form

  $$\forall n_1 \, \exists n_2 \, \forall n_3 \, \cdots \, Q_k n_k \, \Theta,$$

  where there are k alternating number quantifiers beginning with $\forall$ and Θ is Δ^0_0. Here Q_k is $\exists$ if k is even, $\forall$ if k is odd.
- Φ is called Σ^1_k, for $k \in \omega$, if Φ is equivalent to a formula of the form

  $$\exists X_1 \, \forall X_2 \, \exists X_3 \, \cdots \, Q_k X_k \, \Theta,$$

  where Θ is arithmetical and there are exactly k alternating set quantifiers. Here Q_k is $\forall$ if k is even, $\exists$ if k is odd.

- Φ is called Π^1_k, for $k \in \omega$, if Φ is equivalent to a formula of the form

$$\forall X_1 \, \exists X_2 \, \forall X_3 \, \cdots \, Q_k X_k \, \Theta,$$

where Θ is arithmetical and there are exactly k alternating set quantifiers. Here Q_k is $\exists$ if k is even, $\forall$ if k is odd.

A *subsystem* of Z_2 is an L_2 theory whose axioms are logical consequences of the axioms of Z_2 (see Definition 3.1.2). All the subsystems we consider will include the basic axioms of Z_2 and the axioms of classical logic. We now survey the following subsystems of Z_2, listed in increasing order of logical strength: RCA_0, WKL_0, ACA_0, ACA_0^+, ATR_0, Π^1_1-CA_0, and Π^1_2-CA_0.

The subsystem RCA_0

The subsystem RCA_0 consists of the basic axioms, those instances of the comprehension scheme with formulas that are equivalent to both Σ^0_1 and Π^0_1 formulas (the Δ^0_1 *comprehension scheme*), and those instances of the induction scheme with Σ^0_1 formulas. RCA is an abbreviation for "Recursive Comprehension Axiom." See [Sim99, Section I.7] for a precise definition of RCA_0.

Proposition 3.1.6. RCA_0 proves the following:

1. The exponential function $\langle m, n \rangle \mapsto m^n$ is defined for all $m, n \in \mathbb{N}$. There are infinitely many primes and every number is a unique finite product of powers of primes.

2. Every infinite subset of $\mathbb{N}$ is the range of an injective function from $\mathbb{N}$ to $\mathbb{N}$. We call such a function an *enumeration* of the set.

3. For each set $X \subseteq \mathbb{N}$ and each $n \in \mathbb{N}$ the set $(X)_n = \{m \mid 2^n 3^m \in X\}$ exists. We may view X as a code for a sequence of open sets $\langle X_n \mid n \in \mathbb{N} \rangle$, where $X_n = (X)_n$ for each $n \in \mathbb{N}$. In this way, it is possible to quantify over sequences of sets.

Definition 3.1.7. (RCA_0) We define a *function from* $\mathbb{N}$ *to* $\mathbb{N}$ to be a subset f of $\{2^n 3^m \mid n, m \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$ there is exactly one $m \in \mathbb{N}$ with $2^n 3^m \in f$; we let $f(n)$ denote this element m.

Simpson [Sim99, Section II.4] has given definitions of the number systems $\mathbb{Q}$ and $\mathbb{R}$ in RCA_0, which we follow in this thesis. We summarize these definitions here. Elements of $\mathbb{Q}$ are represented as ordered pairs of natural numbers, with a natural equivalence relation. A real number is represented by a sequence $\langle q_i \rangle \subseteq \mathbb{Q}$ such that $|q_i - q_j| < 2^{-i}$ whenever $i < j$.

Definition 3.1.8. (RCA_0) A *countable pseudometric space* consists of a nonempty set $A \subseteq \mathbb{N}$ and a sequence d of real numbers indexed by pairs of elements of A; this sequence is viewed as a function $d\colon A \times A \to \mathbb{R}$. We require that $d(a, a) = 0$, $d(a, b) = d(b, a)$, and $d(a, b) \leq d(a, c) + d(c, b)$ for all $a, b, c \in A$.

A *strong Cauchy sequence* on a countable pseudometric space $\langle A, d\rangle$ is a sequence $\langle a_i \mid i \in \mathbb{N}\rangle \subseteq A$ such that $d(a_i, a_j) < 2^{-i}$ whenever $i \leq j$.

Definition 3.1.9. (RCA_0) A *complete separable metric space* $\widehat{A}$ is defined to consist of all the strong Cauchy sequences on a countable pseudometric space A. For $z = \langle a_i\rangle$ and $z' = \langle a'_i\rangle$ in $\widehat{A}$, we let $d(z, z')$ be the real number given by the strong Cauchy sequence $\langle d(a_{i+2}, a'_{i+2}) \mid i \in \mathbb{N}\rangle$. We define an equality relation on $\widehat{A}$ by letting $z = z'$ if and only if $d(z, z') = 0$.

Simpson [Sim99, Section II.5] has shown that many properties of complete separable metric spaces may be established in RCA_0.

The subsystem WKL_0

In RCA_0, we define $2^{<\mathbb{N}}$ to be the set of all finite sequences of elements of $\{0, 1\}$ and $\mathbb{N}^{<\mathbb{N}}$ to be the set of all finite sequences of elements of $\mathbb{N}$. We say that $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a *tree* if whenever $\sigma \in T$ and τ is an initial segment of σ we have $\tau \in T$. If f is a function from $\mathbb{N}$ to $\mathbb{N}$ and $n \in \mathbb{N}$, we write $f[n]$ for the sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$. We say that f is a *path* through T if $f[n] \in T$ for all $n \in \mathbb{N}$. A tree T is *finitely branching* if for all $\sigma \in T$ of length n there are only finitely many $\tau \in T$ of length $n + 1$ extending σ. A classical theorem known as König's Lemma states that any infinite finitely branching tree has a path.

The subsystem WKL_0 contains RCA_0 and a weak form of König's Lemma which says that if T is an infinite subtree of $2^{<\mathbb{N}}$ then there is a path through T. See [Sim99, Section I.10] for a precise definition of WKL_o.

The subsystem ACA_0

The subsystem ACA_0 consists of the basic axioms, those instances of the comprehension scheme with arithmetical formulas, and those instances of the induction scheme with arithmetical formulas. ACA is an abbreviation for "Arithmetical Comprehension Axiom." See [Sim99, Section I.3] for a precise definition of ACA_0.

In this thesis, we will show that various theorems imply ACA_0 over RCA_0. To establish such results, it will be convenient to have a single L_2-sentence which implies ACA_0 over RCA_0. The next theorem provides such a sentence.

Theorem 3.1.10. ACA_0 is equivalent over RCA_0 to the proposition that whenever f codes a function from $\mathbb{N}$ to $\mathbb{N}$ the set $\{m \mid \exists n[f(n) = m]\}$ (that is, the range of f) exists.

The subsystems ACA_0^+ and ATR_0

The subsystem ACA_0^+ consists of ACA_0 along with an axiom scheme which states than an arithmetically defined functional $P(\mathbb{N}) \to P(\mathbb{N})$ may be iterated along $\mathbb{N}$. A precise definition of ACA_0^+ is given in [Sim99, Section X.3] (see also [BHS87]).

The subsystem ATR_0 consists of ACA_0 along with an axiom scheme which states than an arithmetically defined functional $P(\mathbb{N}) \to P(\mathbb{N})$ may be iterated along any countable well-ordering of $\mathbb{N}$. See [Sim99, Section I.11] for a precise definition of ATR_0.

Theorem 3.1.11. ATR_0 proves the Σ^1_1 *choice scheme*, which consists of the universal closure of each formula of the form

$$\forall n\, \exists X\, \Phi(n, X) \Rightarrow \exists Z\, \forall n\, \Phi(n, (Z)_n),$$

in which Φ is a Σ^1_1 formula.

Proof. See [Sim99, Theorem V.8.3]. □

The subsystems Π^1_1-CA_0 and Π^1_2-CA_0

The subsystem Π^1_1-CA_0 consists of the basic axioms, those instances of the comprehension scheme with Π^1_1 formulas, and those instances of the induction scheme with Π^1_1 formulas. A precise definition of Π^1_1-CA_0 is given in [Sim99, Section I.5].

We will use the following equivalence to show that certain theorems about poset spaces are equivalent to Π^1_1-CA_0 over RCA_0.

Theorem 3.1.12. The following are equivalent over RCA_0:

1. Π^1_1-CA_0
2. For each sequence of trees $\langle T_k \mid k \in \mathbb{N}\rangle$, $T_k \subseteq \mathbb{N}^{<\mathbb{N}}$, there is a set X such that $\forall k\, [k \in X \Leftrightarrow T_k$ has a path$]$.

Proof. See [Sim99, Lemma VI.1.1]. □

The subsystem Π^1_2-CA_0 consists of the basic axioms, those instances of the comprehension scheme with Π^1_2 formulas, and those instances of the induction scheme with Π^1_2 formulas. A precise definition of Π^1_2-CA_0 is given in [Sim99, Section I.5].

Theorem 3.1.13. Π^1_2-CA_0 proves the Σ^1_2 *choice scheme*, which consists of the universal closure of each formula of the form

$$\forall n \, \exists X \, \Phi(n, X) \Rightarrow \exists Z \, \forall n \, \Phi(n, (Z)_n).$$

in which Φ is a Σ^1_2 formula.

Proof. See [Sim99, Theorem VII.6.9], where a sharper result is obtained. □

3.2 Formalization in second-order arithmetic

In this section, we develop the properties of poset spaces (as expounded in Chapter 2) in second-order arithmetic. The definitions in this section are intended to be formalized in RCA_0.

3.2.1 Poset spaces

Definition 3.2.1. A *countable poset* is a pair $\langle P, R \rangle$, where $P \subseteq \mathbb{N}$ and $R \subseteq \{2^p 3^q \mid p, q \in P\}$, such that the binary relation $\preceq$ on P defined by letting $p \preceq q$ if and only if $2^p 3^q \in R$ makes P a partially ordered set.

The sets $\mathrm{UF}(P)$ and $\mathrm{MF}(P)$ cannot be defined directly in second-order arithmetic, because these sets are third-order objects. Instead of defining these sets directly, we will define what it means for a single subset of $\mathbb{N}$ to code an element from one of these sets. (A similar situation arises in ZFC set theory when proper classes such as the ordinals are identified with the extensions of their defining formulas.) To simplify our proofs, we will represent filters by linearly ordered sets that generate them (see Section 4.1 for a justification of this choice). We fix an L_2 formula which defines the relation $p \in \mathrm{ucl}(U)$ for $p \in P$ and $U \subseteq P$:

$$p \in \mathrm{ucl}(U) \quad \equiv \quad \exists q \in U \, [q \preceq p].$$

We do not assume that $\mathrm{ucl}(U)$ exists when we write $p \in \mathrm{ucl}(U)$. We also fix an L_2 formula $\mathrm{LO}(U)$ which states that U is a linearly ordered subset of P:

$$\mathrm{LO}(U) \quad \equiv \quad \forall q \in U \, \forall r \in U [q \in P \wedge r \in P \wedge (q \preceq r \vee r \preceq q)].$$

Definition 3.2.2. We now define which subsets of $\mathbb{N}$ code points in poset spaces:

$$\begin{aligned} x \in \mathrm{UF}(P) &\equiv \mathrm{LO}(x) \wedge \forall q \in P\, \exists r \in x[q \not\prec r] \\ x \in \mathrm{MF}(P) &\equiv \mathrm{LO}(x) \wedge \forall y\, [(\mathrm{LO}(y) \wedge x \subseteq \mathrm{ucl}(y)) \Rightarrow y \subseteq \mathrm{ucl}(x)] \end{aligned}$$

We have thus defined $\mathrm{MF}(P)$ and $\mathrm{UF}(P)$ to consist of those linearly ordered subsets of P whose upward closures correspond to the correct type of filter. Each filter may have many linearly ordered subsets that generate it; we say that two elements x, y of $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$ are *equal*, and write $x = y$, if $x \subseteq \mathrm{ucl}(y)$ and $y \subseteq \mathrm{ucl}(x)$.

Remark 3.2.3. The definition of equality for points of a countably based poset space is similar to the definition of equality for points of complete separable metric spaces in [Sim99, Section II.5], which we now recall. Two Cauchy sequences $z = \langle a_i \rangle$ and $z' = \langle a'_i \rangle$ are said to be equal if $d(z, z') = 0$. If z and z' are strong Cauchy sequences, then $z = z'$ if and only if $\forall i[d(a_i, a'_i) < 2^{-(i+1)}]$. There is a thus a Π^0_1 formula which tells if points of $\widehat{A}$ are equal. The formula which tells if points of a countably based poset space are equal is Π^0_2 but not Π^0_1.

We code a basic open set N_p with the number p. If x is in $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$ then $x \in N_p$ if and only if $p \in \mathrm{ucl}(x)$; so we can easily tell if a coded point is in a basic open set.

If P is a countable poset then every open subset of $\mathrm{UF}(P)$ (or $\mathrm{MF}(P)$) is a countable union of basic open sets. We are thus justified in defining a *coded open set* to be a subset of P; each $U \subseteq P$ codes the open set $N_U = \bigcup_{p \in U} N_p$. A point x is in the open set coded by U if and only if there is an $r \in x$ such that $r \preceq p$ for some $p \in U$. Thus the relation which tells if a coded point is in a coded open set is Σ^0_1.

We now present several theorems to justify our choice of coding for $\mathrm{MF}(P)$ and $\mathrm{UF}(P)$. The first theorem shows that RCA_0 proves that $\mathrm{UF}(P)$ is nonempty, and ACA_0 proves that $\mathrm{MF}(P)$ is nonempty.

Theorem 3.2.4. Let P be a countable poset and $p \in P$. RCA_0 proves that there is an element of $\mathrm{UF}(P)$ in N_p, and ACA_0 proves that there is an element of $\mathrm{MF}(P)$ in N_p.

Proof. Write $P = \langle p_i \mid i \in \mathbb{N} \rangle$. We construct an unbounded descending sequence $Q = \langle q_i \rangle$ below p by induction. Let $q_1 = p$. Given q_i, decide if $p_i \prec q_i$. If so, let $q_{i+1} = p_i$. Otherwise, let $q_{i+1} = q_i$. Clearly $\langle q_i \rangle$ is an unbounded descending sequence. The set Q is Δ^0_1; in order for p_j to be in Q it must be that $p_j = q_{j+1}$. Thus RCA_0 proves that Q exists.

To construct an element of $\mathrm{MF}(P)$ in N_p, we first use arithmetic comprehension to form the set $A = \{\langle p, q\rangle \mid p \perp q\}$. We then construct a descending sequence by induction. Let $q_1 = p$. Given q_i, we use A to decide if $q_i \perp p_i$. If so, we let $q_{i+1} = q_i$. Otherwise, we let $q_{i+1} = p_k$, where k is the least number such that p_k is a common extension of q_i and p_i. Because $Q = \langle q_i\rangle$ is eventually below or incompatible with every element of P, the upward closure of Q is a maximal filter. Thus $Q \in \mathrm{MF}(P)$. It is immediate that Q is definable by an arithmetic formula. □

Corollary 3.2.5. In RCA_0, we can prove that $N_p \cap N_q = \varnothing$ in the sense of $\mathrm{UF}(P)$ if and only if $p \perp q$. We can prove the corresponding result for $\mathrm{MF}(P)$ in ACA_0.

Proof. If $N_p \cap N_q \neq \varnothing$ then there is a descending sequence $\langle p_i\rangle$ which is eventually below p and eventually below q. Thus, for large enough k, p_k is a common extension of p and q. Thus RCA_0 proves that $N_p \cap N_q \neq \varnothing$ implies $p \not\perp q$, for both $\mathrm{MF}(P)$ and $\mathrm{UF}(P)$.

For the converse, suppose $r \preceq p$ and $r \preceq q$. RCA_0 proves N_r is nonempty in the sense of $\mathrm{UF}(P)$; hence $N_p \cap N_q \neq \varnothing$ in $\mathrm{UF}(P)$. Similarly, ACA_0 proves N_r is nonempty in $\mathrm{MF}(P)$. □

Corollary 3.2.6. Let P be a countable poset. RCA_0 proves that $\mathrm{UF}(P)$ has a countable dense subset. ACA_0 proves that $\mathrm{MF}(P)$ has a countable dense subset.

Proof. Let X be $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$. Because the proof of Theorem 3.2.4 is uniform, we may follow that proof to construct a sequence $\langle F_p \mid p \in P\rangle \subseteq X$ such that $F_p \in N_p$ for $p \in P$. This construction may be performed in RCA_0 for UF spaces, and in ACA_0 for MF spaces. □

Recall that a topological space has the property of Baire if every intersection of countably many dense open sets is dense.

Theorem 3.2.7. Let P be a countable poset. RCA_0 proves that $\mathrm{UF}(P)$ has the property of Baire, and ACA_0 proves that $\mathrm{MF}(P)$ has the property of Baire.

Proof. The proof is essentially the same as Theorem 3.2.4. We only need to ensure that our descending sequence meets a countable collection of dense open sets (that is, we must ensure the descending sequence is eventually inside each open set in the collection). We can do this by breaking the construction into odd and even stages. At stage $2n$ we perform the action from stage n of the proof of Theorem 3.2.4. At stage $2n+1$ we meet the nth dense open set. □

Open Problem 3.2.8. Does RCA_0 prove that $\mathrm{MF}(P)$ is nonempty? This question can be rephrased as a question of effective mathematics: Is there a computable poset P such that for every r.e. linearly ordered set $X \subseteq P$ there is an r.e. linearly ordered set $Y \subseteq P$ such that $\mathrm{ucl}(X) \subsetneq \mathrm{ucl}(Y)$? It can be shown that such a poset must have no minimal elements and the relation $p \perp q$ cannot be computable.

This question is also relevant to a possible reversal of Theorem 3.2.7 for $\mathrm{MF}(P)$. It is not hard to show that "$\mathrm{MF}(P)$ has the property of Baire" implies "For every $p \in P$, there is an element of $\mathrm{MF}(P)$ in N_p" over RCA_0.

3.2.2 Continuous functions

In ZFC, a function f from a space X to a space Y is defined to be a subset of $X \times Y$ such that for each $x \in X$ there is exactly one $y \in Y$ with $\langle x, y\rangle \in f$; this y is denoted $f(x)$. Thus a function is a third-order object which may not be representable in second-order arithmetic if X or Y is uncountable. If f is a continuous function, however, we may approximate $f(x)$ arbitrarily well using only sufficiently accurate approximations of x. We use this fact to define codes for continuous functions on poset spaces.

Definition 3.2.9. (RCA_0) Let P and Q be countable posets. Let X be $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$, and let Y be $\mathrm{MF}(Q)$ or $\mathrm{UF}(Q)$. A *code for a continuous function* is a subset of $\mathbb{N} \times P \times Q$. Each code F for a continuous function induces a partial function f from X to Y, defined by letting $f(x) = \{q \in Q \mid \exists n \in \mathbb{N}\, \exists p \in x\, [\langle n, p, q\rangle \in F]\}$ whenever this set is a point in Y. We will only be concerned with codes that induce total functions from X to Y.

This abstract definition requires justification. We will show, in ZFC, that every continuous function may be represented. We will show in ACA_0 that the preimage of an open set under a coded continuous function is an open set. RCA_0 is strong enough to evaluate coded continuous functions into the poset representation of the real numbers. The parameter n in a condition $\langle n, p, q\rangle$ allows composition of functions to be carried out in RCA_0; see Lemma 4.5.18. These results indicate that our definition of a code for a continuous function is acceptable. We discuss additional motivations for our choice of coding at the end of this section.

It is clear from the way we have coded continuous functions that if a point maps into an open set then there is an open neighborhood of the point which maps into the open set. We use this fact to show, in ACA_0, that every preimage of an open set under a coded continuous function is open.

Proposition 3.2.10. (ACA_0) Suppose that P and Q are countable posets, X is $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$, and Y is $\mathrm{MF}(Q)$ or $\mathrm{UF}(Q)$. Let $F \subseteq \mathbb{N} \times P \times Q$ be a code for a continuous function $f\colon X \to Y$. For every $V \subseteq Q$ there is a $U \subseteq P$ such that $N_U = f^{-1}(N_V)$.

Proof. Let $U = \{p \in P \mid \exists n \in \mathbb{N}\, \exists q \in V\, [\langle n, p, q\rangle \in F]\}$. It is clear that U has the desired property. □

We now consider the problem of which functions may be encoded. After considering some basic examples, we show in ZFC that every continuous function has a code.

Proposition 3.2.11. (RCA_0) Let P and Q be countable posets, let X be $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$, and let Y be $\mathrm{UF}(Q)$ or $\mathrm{MF}(Q)$. There is a coded continuous function $i\colon X \to X$ with $i(x) = x$ for all $x \in X$. For each $y \in Y$ there is a coded continuous function $c_y\colon X \to Y$ such that $c_y(x) = y$ for all $x \in X$.

Proof. The set $\{\langle 0, p, p\rangle \mid p \in P\}$ codes the identity function i, while the set $\{\langle 0, p, q\rangle \mid p \in P, q \in y\}$ codes the constant function c_y. Each of these sets is definable in RCA_0. Is straightforward to verify that each of these sets is a code for the correct continuous function. □

Proposition 3.2.12. (ZFC) Let P and Q be countable posets. Let X be $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$, and let Y be $\mathrm{UF}(Q)$ or $\mathrm{MF}(Q)$. Let $f\colon X \to Y$ be a continuous function. Then there is a code F for a continuous function such that F induces f.

Proof. Let $F = \{\langle 0, p, q\rangle \in P \times Q \mid f(N_p) \subseteq N_q\}$. We will show that F induces f. Let $x \in X$ be fixed and let $f(x) = y$. We must show $F[x] = y$. Fix $q \in y$; since f is continuous, there is some p with $x \in N_p$ such that $f(N_p) \subseteq N_q$. Thus $\langle 0, p, q\rangle \in F$, so $q \in F[x]$. Conversely, if $q \in F[x]$ then there is some p, $x \in N_p$, such that $\langle 0, p, q\rangle \in F$. This implies $f(N_p) \subseteq N_q$, hence $y \in N_q$. Thus $q \in y$. □

We next show that coded continuous functions may be composed in RCA_0. The proof of this result uses the number parameter in the conditions used to code continuous functions.

Lemma 3.2.13. (RCA_0) Let X, Y, Z be poset spaces based on the countable posets P, Q, R, respectively. Suppose that F is a code for a continuous function $f\colon X \to Y$ and G is a code for a continuous function $g\colon Y \to Z$. Then there is a code H for the continuous function $g \circ f\colon X \to Z$.

Proof. Let $\{q_i\}$ be an enumeration of Q. We define H to be the set of all $\langle n, p, r\rangle \in \mathbb{N} \times P \times R$ such that there are m_1, m_2, i, all less than n, such that $\langle m_1, p, q_i\rangle \in F$ and $\langle m_2, q_i, r\rangle \in G$. The set H may be formed by Δ^0_1 comprehension. It is immediate that H is a code for $g \circ f$. □

Remark 3.2.14. We compare our definition of continuous functions between countably based poset spaces with the definition of codes for continuous functions between complete separable metric spaces given in [Sim99, Section II.6]. We summarize this definition. Let $\langle \widehat{A}, d_A\rangle$ and $\langle \widehat{B}, d_B\rangle$ be complete separable metric spaces. A *code for a continuous function* from $\widehat{A}$ to $\widehat{B}$ is defined in RCA_0 to be a subset F of $\mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ such that

1. If $\langle n, a, r, b, s\rangle \in F$ and $d_A(a', a) + r' < r$ then $\langle m, a', r', b, s\rangle \in F$ for some $m \in \mathbb{N}$.

2. If $\langle n, a, r, b, s\rangle \in F$ and $d_B(b, b') + s < s'$ then $\langle m, a, r, b', s'\rangle \in F$ for some $m \in \mathbb{N}$.

3. If $\langle n, a, r, b, s\rangle \in F$ and $\langle m, a, r, b', s'\rangle \in F$ then $d_B(b, b') < s + s'$.

4. For every Cauchy sequence $z \in \widehat{A}$, there is exactly one point of $\widehat{B}$ in
$$\bigcap \bigl\{ B_{\widehat{B}}(b, s) \mid \exists n\, \exists a\, \exists r\, [x \in B_{\widehat{A}}(a, r) \wedge \langle n, a, r, b, s\rangle \in F] \bigr\}.$$

In light of this definition, it may seem natural to define a code for a continuous function between poset spaces X and Y (based on countable posets P and Q, respectively) to be a subset F of $\mathbb{N} \times P \times Q$ such that

1. If $\langle n, p, q\rangle \in F$ and $p' \preceq_P p$ then $\langle m, p', q\rangle \in F$ for some $m \in \mathbb{N}$.

2. If $\langle n, p, q\rangle \in F$ and $p \preceq_Q q'$ then $\langle m, p, q'\rangle \in F$ for some $m \in \mathbb{N}$.

3. If $\langle n, p, q\rangle \in F$ and $\langle m, p', q\rangle \in F$ then for each $p'' \in P$ with $p'' \preceq p$ and $p'' \preceq p'$ there is a $q'' \in Q$ such that $q'' \preceq q$, $q'' \preceq q'$, and $\langle o, p'' q''\rangle \in F$ for some $o \in \mathbb{N}$.

4. For each point $x \in X$, the set $F[x] = \{q \in Q \mid \exists n \in \mathbb{N}\, \exists p \in x\, [\langle n, p, q\rangle\}$ is an element of Y.

We do not know whether every continuous function has a code which satisfies this definition. We have chosen the weaker definition of coded continuous functions presented in Definition 3.2.9 because we are able to show (in ZFC) that every continuous function has a code (Proposition 3.2.12).

3.2.3 Homeomorphisms and metrizability

Definition 3.2.15. (RCA_0) Let X and Y be countably based poset spaces and let h be a continuous function from X to Y. We say that h is *open* if the image of each open set in X is open in Y, that is, for every open set $U \subseteq X$ there is an open set $V \subseteq Y$ such that $x \in U \Leftrightarrow h(x) \in V$ for all $x \in X$. The function h is *weakly open* if the image of each basic open set in X is an open set in Y.

The function h is *strongly open* if there is an arithmetical functional Φ such that for every open $U \subseteq X$ the set $\Phi(U)$ is open in Y and $f(U) = \Phi(U)$. (We may not be able to form the set $\Phi(U)$ in RCA_0.)

The following proposition follows directly from definitions.

Proposition 3.2.16. ACA_0 proves that every strongly open map is open. RCA_0 proves that every open map is weakly open.

Open Problem 3.2.17. Determine the Reverse Mathematics strengths of the following propositions.

1. Every continuous open map between countably based MF spaces is strongly open.
2. Every continuous map between countably based MF spaces which is weakly open is open.

The same questions may be asked about UF spaces. Does the Reverse Mathematics strength of these propositions change if we only consider continuous surjections, continuous injections, or continuous bijections?

Definition 3.2.18. (RCA_0) We say that X and Y are *homeomorphic* if there is a coded continuous bijection $h\colon X \to Y$ with a coded continuous inverse h^{-1}.

Proposition 3.2.19. (RCA_0) Every homeomorphism between countably based poset spaces is strongly open.

Proof. Let X and Y be poset spaces based on the countable posets P and Q, respectively. Let $h\colon X \to Y$ be given along with the inverse map h^{-1}.

Let H^{-1} be the code for h^{-1}. Then, for any coded open $U \subseteq X$, we have

$$h(U) = (h^{-1})^{-1}(U) = \{q \in Q \mid \exists p \in P \, \exists n \in \mathbb{N} \, [\langle n, p, q \rangle \in H^{-1}\}. \qquad \square$$

It is important to note that the proposition that X is homeomorphic to Y is, apparently, not equivalent in RCA_0 to the proposition that there is a a continuous open bijection from X to Y.

Open Problem 3.2.20. Determine the Reverse Mathematics strength of the following propositions. Every continuous open bijection between countably based MF spaces has a continuous inverse. Every continuous open bijection between countably based UF spaces has a continuous inverse. Similar questions may be asked when one space is a countably based MF space and the other is a countably based UF space. Similar questions may be asked concerning continuous strongly open bijections.

Definition 3.2.21. (ACA_0) Let $\widehat{A}$ be a complete separable metric space. We define the *standard poset representation* of $\widehat{A}$ to be the poset $P_{\widehat{A}} = A \times \mathbb{Q}^+$, ordered such that $\langle a, r \rangle \prec \langle a', r' \rangle$ if and only if $d(a, a') + r < r'$.

The proof of the next proposition is a straightforward formalization of the proof of Theorem 2.3.9.

Proposition 3.2.22. (ACA_0) Let $\widehat{A}$ be a complete separable metric space. There is a canonical arithmetical bijection between Cauchy sequences in $\widehat{A}$ and maximal filters on $P_{\widehat{A}}$. Moreover, $\mathrm{UF}(P_{\widehat{A}}) = \mathrm{MF}(P_{\widehat{A}})$.

Definition 3.2.21 is made in ACA_0 only because it is not clear that the order relation on $P_{\widehat{A}}$ may be formed in RCA_0 for an arbitrary complete separable metric space $\langle \widehat{A}, d \rangle$. In some cases, RCA_0 is able to define $P_{\widehat{A}}$ and its order relation. In particular, we may form the standard poset representations of $\mathbb{R}$ and $[0, \infty)$ in RCA_0. RCA_0 is able to prove Proposition 3.2.22 with the added assumption that $P_{\widehat{A}}$ exists.

Open Problem 3.2.23. Does RCA_0 prove that for every complete separable metric space X there is a canonical poset P such that there is an arithmetical bijection between the points of X and the points of $\mathrm{MF}(P)$? Can we assume $\mathrm{MF}(P) = \mathrm{UF}(P)$?

Theorem 2.3.9 shows, in ZFC, that $\mathrm{MF}(P_{\widehat{A}}) \cong \widehat{A}$ and $\mathrm{MF}(P_{\widehat{A}}) = \mathrm{UF}(P_{\widehat{A}})$. For the reset of this thesis, we will identify each complete separable metric space $\langle \widehat{A}, d \rangle$ with its standard poset representation.

Definition 3.2.24. (ACA_0) A countably based poset space X is said to be *homeomorphic to a complete separable metric space* if there is a complete separable metric space $\widehat{A}$ such that X is homeomorphic to $\mathrm{MF}(P_{\widehat{A}})$. This means, by Definition 3.2.18, that there is a coded continuous bijection from X to $\mathrm{MF}(P_{\widehat{A}})$ with a coded continuous inverse.

Definition 3.2.25. (RCA_0) A countably based poset space X is *metrizable* if there is a continuous function $d\colon X \times X \to [0, \infty)$ such that

1. d is a metric. That is, d is symmetric, $d(x, y) = 0$ if and only if $x = y$, and d satisfies the triangle inequality.
2. For each $U \subseteq P$ there is a set $\{\langle x_i, r_i\rangle \mid i \in \mathbb{N}\} \subseteq X \times \mathbb{Q}^+$ such that $\bigcup_{p\in U} N_p = \bigcup B(x_i, r_i)$. Here, $B(x_i, r_i) = \{x \in X \mid d(x, x_i) < r_i\}$.
3. For each $\{\langle x_i, r_i\rangle \mid i \in \mathbb{N}\} \subseteq X \times \mathbb{Q}^+$ there is a set $U \subseteq P$ such that $\bigcup B(x_i, r_i) = \bigcup_{p\in U} N_p$.

We say that d is *compatible with the original topology* if and only if conditions (2) and (3) hold. These conditions imply that the topology on X induced by d is the same as the poset topology on X.

A countably based poset space is *completely metrizable* if there is a metric d compatible with the topology on X such that for every strong Cauchy sequence $\langle x_i \mid i \in \mathbb{N}\rangle$ of points in X there is an $x \in X$ such that $\langle x_i\rangle$ converges to X in the poset topology. It is clear that this may be expressed as a sentence in the language of second-order arithmetic.

Note that the definition of metrizability requires that each set which is open in the poset topology must be open in the metric topology, but does not give a uniform method of converting a code for an open subset in the poset topology to an open subset in the metric topology. Similarly, no uniform method for converting open sets in the metric topology to open sets in the poset topology is given. Lemmas 4.3.9 and 4.3.10 investigate the problem of establishing such uniformity. We now define a weaker form of metrizability, which is equivalent to metrizability over ZFC.

Definition 3.2.26. (RCA_0) A countably based poset space X is *weakly metrizable* if there is a function $d\colon X \times X \to [0, \infty)$ such that

1. d is a metric.
2. For every $p \in P$ and every $x \in N_p$ there is an $r \in \mathbb{Q}^+$ such that $B(x, r) \subseteq N_p$.
3. For every $x \in X$ and every $r \in \mathbb{Q}^+$ there is a $p \in P$ such that $x \in N_p$ and $N_p \subseteq B(x, r)$.

Open Problem 3.2.27. Determine the Reverse Mathematics strength of the proposition that every weakly metrizable countably based MF space is metrizable. The same question may be asked for countably based UF spaces.

There is a natural definition of a *completely weakly metrizable* poset space, which is obtained by replacing metrizability by weak metrizability in the definition of a completely metrizable poset space. What is the Reverse Mathematics strength of the proposition that every weakly completely metrizable countably based MF space is completely metrizable? What is the strength of the proposition that every completely weakly metrizable countably based MF space is metrizable? Similar questions may be asked about countably based UF spaces.

A key property that a method for coding continuous functions must possess is that RCA_0 must be able to evaluate functions into the real numbers. That is, if f is a continuous function from a space X into the real numbers, $x \in X$, and $z \in \widehat{\mathbb{Q}}$ is a strong Cauchy sequence, then the predicates $f(x) < z$ and $f(x) = z$ must be definable by formulas of low complexity. The next lemma shows, in RCA_0, that the method we have chosen to code continuous is satisfactory.

Lemma 3.2.28. (RCA_0) Let P be a countable poset and let X be UF(P) or MF(P). Let Y be the standard poset representation of $\mathbb{R}$ or $[0, \infty)$, and let F be a code for a continuous function $f\colon X \to Y$. There is a Σ^0_1 formula $\Phi(x, z, F)$ with the free variables shown such that if z is a strong Cauchy sequence of rationals and $x \in X$ then $\Phi(x, z, F)$ holds if and only if $f(x) < z$. There is a Π^0_1 formula $\Theta(x, z, F)$ which holds if and only if $f(x) = z$.

Proof. Let F, x, and r be as in the statement. Let $\langle n_i, p_i, \langle q_i, r_i\rangle\rangle$ be an enumeration of all of the conditions $\langle x, p, \langle q, r\rangle\rangle$ in F such that $x \in N_p$. We define by induction a sequence $\langle s_i \rangle$ of rationals. To define s_i, we choose the least j such that $r_i < 2^{-i}$, and let $s_i = q_j$. Because F codes a continuous function from X to Y, such a j will always exist, and the sequence $\langle s_i \rangle$ will be a strong Cauchy sequence converging to $f(x)$. It is well known that the relation $z < z'$ is Σ^0_1 and the relation $z = z'$ is Π^0_1 for strong Cauchy sequences z and z'. The lemma follows. □

Chapter 4

Reverse Mathematics of Poset Spaces

In this chapter, we prove Reverse Mathematics results for poset spaces using the formalization of poset spaces presented in Chapter 3.

4.1 Filter extension theorems

In this section, we investigate how difficult it is to extend a given linearly ordered subset or filter in a countable poset P to an element of $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$. We will see that RCA_0 can construct unbounded descending sequences, but RCA_0 is not strong enough to construct the upward closures of arbitrary subsets of P. These results justify our choice of encoding $\mathrm{UF}(P)$ and $\mathrm{MF}(P)$ as descending sequences whose upward closure is the appropriate kind of filter. If we were to define poset spaces to consist of filters, rather than descending sequences, then it would be more difficult to work with poset spaces in RCA_0.

We first note that RCA_0 proves that if $X \subseteq P$ is linearly ordered then there is a descending sequence $\langle p_i \rangle \subseteq X$ which is cofinal in X. Thus, there is no loss in generality if we consider descending sequences rather than linearly ordered subsets of P. We say that a descending sequence X *extends* to a sequence Y if $X \subseteq \mathrm{ucl}(Y)$.

Lemma 4.1.1. RCA_0 proves that every descending sequence extends to an unbounded descending sequence.

Proof. Let X be a descending sequence on P. If X is unbounded then we are done; otherwise there is some $p \in P$ which is a lower bound for X. We may construct an unbounded descending sequence below p; any such sequence will extend X. $\square$

We remark that the proof of Lemma 4.1.1 is nonuniform in a certain sense; RCA_0 would not prove a uniform version of the lemma.

We now show that we may safely work with filters (the upward closures of descending sequences) in ACA_0, but we cannot take the upward closure of arbitrary sets in RCA_0. Thus the definition of $\mathrm{UF}(P)$ as a class of descending sequences is appropriate for RCA_0. When we are working over ACA_0, we may pass freely between filters and the descending sequences that generate them.

Theorem 4.1.2. The following are equivalent over RCA_0:

1. ACA_0.
2. Every subset of a countable poset has an upward closure.
3. Every linearly ordered subset of a countable poset has an upward closure.

Proof. (1) implies (2): If $X \subseteq P$, then $\mathrm{ucl}(X) = \{p \in P \mid \exists q \in X[q \preceq p]\}$. Thus $\mathrm{ucl}(X)$ exists by arithmetical comprehension.

Clearly (2) implies (3). To finish the proof, we show (3) implies (1). We will use the criterion stated in Theorem 3.1.10. Let f be a function from $\mathbb{N}$ to $\mathbb{N}$; we must show that the range of f exists.

Let $P = \{2^k \mid k \in \mathbb{N}\} \cup \{3^j \mid j \in \mathbb{N}\}$. Define a relation $\prec$ on P by letting $2^j \prec 2^i$ whenever $j > i$, $3^i | 3^j$ for all i, j, and $2^i \prec 3^j$ if and only if there is some $k \leq i$ such that $f(k) = j$. The order relation $\prec$ is Δ^0_1 definable, so we may define $\prec$ in RCA_0. It is immediate that $\prec$ is a poset order on P.

Let $X = \{2^k \mid k \in \mathbb{N}\} \subseteq P$. By assumption, $\mathrm{ucl}(X)$ exists. Now $3^j \in \mathrm{ucl}(X)$ if and only if there is some k such that $f(k) = j$; thus the range of f may be formed by Δ^0_1 comprehension relative to $\mathrm{ucl}(X)$. $\square$

The final theorem in this section will show that maximal filters are much more difficult to construct than unbounded filters. We first establish several lemmas.

Lemma 4.1.3. (RCA_0) The following proposition implies ACA_0. Every descending sequence in a countable poset P extends to an element of $\mathrm{MF}(P)$.

Proof. Assume that every descending sequence in a countable poset P extends to an element of $\mathrm{MF}(P)$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be fixed. We will show that the range of f exists.

For each $n \in \mathbb{N}$, define a poset P_n by letting

$$P_n = \{2^n 3^j \mid j \in \mathbb{N}\} \cup \{2^n 5^k \mid \forall l < k\, [f(l) \neq n]\}.$$

The order on P_n is defined as follows:

1. $2^n 3^j \preceq 2n3^k$ if and only if $j \geq k$.

2. $2^n 5^j \preceq 2^n 5^k$ if and only if $j \geq k$.

3. $2^n 5^j \preceq 2^n 3^k$ if and only if $j \geq k$.

4. $2^n 3^j \preceq 2^n 5^k$ never.

Note that each P_n and its order relation are uniformly Δ^0_1 definable relative to f, and thus RCA_0 is able to form the sequence $\langle P_i \rangle$.

For each $n \in \mathbb{N}$, the set $F_n = \{2^n 3^i \mid i \in \mathbb{N}\}$ is a maximal filter on P_n if and only if n in the range of F. Suppose that $f(k) = n$ and k is the least such number. Then for all $l \geq k$ we have $2^n 5^l \notin P_n$, and thus there is no common extension of $2^n 3^{k+1}$ and $2^n 5^0$ in P_n. This means there is no filter extending F_n which contains any element of the form $2^n 5^i$; so F_n is maximal. Now suppose that n is not in the range of f. Then every two elements of P_n have a common extension, and thus F_n is not a maximal filter on P_n. We have shown, in fact, that F_n has a unique extension to a maximal filter (either F_n or P_n) and n is not in the range of f if and only if there is an element of the for $2^n 5^i$ in this extension.

We next form the product poset $P = \widetilde{\prod}_{n\in\mathbb{N}} P_n$ and the product filter $F = \widetilde{\prod}_{n\in\mathbb{N}} F_n$. It is clear that these sets may be formed in RCA_0. Let G be an extension of F to a maximal filter on P. Now for any $n \in \mathbb{N}$, we see that n is not in the range of f if and only if there is a condition in G which contains an element of P_n of the form $2^n 5^i$. Thus the predicate $n \notin \text{range}(f)$ is definable by a Σ^0_1 formula relative to G. The relation $n \in \text{range}(f)$ is trivially Σ^0_1. We may thus form the set $\{n \mid \exists k[f(n) = k]\}$ by Δ^0_1 comprehension. $\square$

Lemma 4.1.4. (Π^1_1-CA_0) Every filter on a countable poset extends to a maximal filter.

Proof. Let X be a descending sequence in a countable poset $P = \langle p_i \rangle$. Let M be a countable coded β-model containing P, the order relation on P, and X. Recall that $M = \langle M_i \rangle$ is a sequence of sets. Define the formula

$$\Phi(X, p) \equiv \exists Y [X \subseteq Y \wedge p \in Y \wedge \text{Filt}(Y, P)],$$

where $\mathrm{Filt}(Y, P)$ is the arithmetical formula which says that Y is a filter on P. Note that Φ is Σ^1_1 and is thus absolute to M.

We will construct a sequence of integers n_i by induction. Choose n_0 such that $M_{n_0} = X$. We will use the induction hypothesis that M_{n_i} is always a filter; this is clearly true for $i = 0$. Π^1_1-CA_0 (actually ATR_0) proves that the satisfaction predicate for M exists; the induction will query this predicate at each stage.

Given n_i, ask whether $\Phi(M_{n_i}, p_i)$ holds in M. If it does not, let $n_{i+1} = n_i$. If $\Phi(M_{n_i}, p_i)$ does hold, then let n_{i+1} be the least number such that $M_{n_{i+1}}$ is a filter extending M_{n_i} and containing p_i.

We have thus constructed a sequence $N = \langle n_i \rangle$ which is arithmetically definable from M and the satisfaction relation on M. Thus Π^1_1-CA_0 is strong enough to construct this sequence. From N, we can obtain an increasing sequence of filters:

$$M_{n_0} \subseteq M_{n_1} \subseteq \cdots \subseteq M_{n_i} \subseteq \cdots .$$

Let $Y = \bigcup_{i \in \mathbb{N}} M_{n_i}$; this set exists by arithmetical comprehension relative to M and N.

It is straightforward to show that Y is a filter, because each M_{n_i} is a filter. Also, $X = M_{n_0} \subseteq Y$. It only remains to show that Y is a maximal filter. Suppose that Y' is a filter with $Y \subseteq Y'$. Choose $p_i \in Y'$. Then Y' is a filter extending M_{n_i} and containing p_i, so $p_i \in M_{n_{i+1}} \subseteq Y$. This shows $Y' \subseteq Y$; so Y is maximal. □

Theorem 4.1.5. The following are equivalent over RCA_0:

1. Π^1_1-CA_0.
2. Every descending sequence in a countable poset P extends to an element of $\mathrm{MF}(P)$.
3. Every filter on a countable poset extends to a maximal filter.

Proof. Lemma 4.1.4 shows that Π^1_1-CA_0 proves (3). It is clear that (2) and (3) are equivalent over ACA_0. Thus Π^1_1-CA_0 proves (2). Moreover, (2) implies (3) over RCA_0, because (as Lemma 4.1.3 shows) (2) implies ACA_0 over RCA_0.

We have thus shown that Π^1_1-CA_0 implies (2) and (2) implies (3) over RCA_0. It remains to show that (3) implies Π^1_1-CA_0 over RCA_0. We will use the fact that Π^1_1-CA_0 is equivalent over RCA_0 to the proposition "for every sequence $\langle T_i \rangle$ of subtrees of $\mathbb{N}^{<\mathbb{N}}$ there is a set N such that $n \in N$ if and only if T_n has no path.

Begin with a sequence of trees $\langle T_n \rangle$. For each tree T_n, form a partial order $\langle P_n, \prec_n \rangle$ consisting of T_n plus a descending sequence $\{a_i^n | i \in \mathbb{N}\}$ disjoint from T_n. The partial ordering on P_n is defined so that $\tau \prec_n a_i^n$ whenever $i < |\tau|$, $\sigma \prec_n \tau$ whenever σ extends τ, and $a_j^n \prec_n a_i^n$ whenever $i < j$. Note that $F_n = \{a_i^n : i \in \mathbb{N}\}$ is a filter on P_n and is maximal if and only if T_n has no path.

We next form the product partial order $P = \widetilde{\prod} P_n$ and the product filter $F = \widetilde{\prod} F_n$. These sets exist by Δ^0_1 comprehension. By assumption, we can extend F to a maximal filter G. Let $K = \{k \mid T_k \text{ has a path }\}$. Now T_n has a path just in case the product condition consisting of $\{a_1^m | m < n\}$ plus the root of T_n is in G. Hence K can be defined by a Δ^0_1 formula relative to G. So RCA_0 proves that K exists. □

4.2 Subspaces and product spaces

In Section 2.3.2, we saw that the class of countably based MF spaces is closed under countable products and under G_δ subspaces. In this section, we investigate the set existence axioms necessary to obtain these results. We will see that RCA_0 can form products of countably based MF spaces, but forming G_δ subspaces sometimes requires Π^1_1-CA_0.

Theorem 4.2.1. RCA_0 proves that if $\langle P_i \mid i \in \mathbb{N} \rangle$ is a sequence of countable posets then there is a countable poset P such that $\mathrm{MF}(P)$ is homeomorphic to the topological product $\prod_i \mathrm{MF}(P_i)$.

Proof. The proof of Theorem 2.3.15 goes through in RCA_0. The product poset, the projection maps, and the map which sends a sequence of filters to a single product filter are all definable by Δ^0_1 formulas. □

A *coded G_δ subspace* of a poset space X is a sequence $\langle U_i \mid i \in \mathbb{N} \rangle$ of open subsets; the sequence represents the G_δ set $\bigcap_{i \in \mathbb{N}} N_{U_i}$.

Lemma 4.2.2. Let U be a coded G_δ subspace of $\mathrm{MF}(P)$, where P is a countable poset. Then Π^1_1-CA_0 proves that there is a countable poset Q such that $U \cong \mathrm{MF}(Q)$.

Proof. We indicate how the proof of Theorem 2.3.18 may be formalized in Π^1_1-CA_0. We may define

$$\begin{aligned} Q = \{\langle n, p \rangle \mid p \in P \wedge \forall i \leq n \, [N_p \subseteq U_i] \\ \wedge \exists F \subseteq P \, [\mathrm{LO}(F) \wedge p \in F \wedge \forall i \, \exists q \in F \, (q \in U_i)]\}. \end{aligned}$$

The defining formula is Σ^1_1; thus Q exists by Π^1_1 comprehension. The order relation on Q is Δ^0_1 definable. It is interesting to note that the definition of Q quantifies over descending sequences which meet each open set U_i rather than quantifying over maximal filters; this trick enables us to avoid Π^1_2 comprehension. We are using the fact that, over Π^1_1-CA_0, a G_δ set $U = \bigcap U_i$ contains a point of $\mathrm{MF}(P)$ if and only if there is a linearly ordered $F \subseteq P$ that meets each open set U_i; see Theorem 4.1.5.

The map $F\colon U \to \mathrm{MF}(Q)$, defined by $x \mapsto \{\langle n, p\rangle \in Q \mid x \in N_p\}$, is represented by the coded continuous function $\{\langle 0, p, \langle n, p\rangle\rangle \mid \langle n, p\rangle \in Q\}$, which may be formed by Δ^0_1 comprehension relative to Q. The inverse to this map is encoded by $\{\langle 0, \langle n, p\rangle, p\rangle \mid \langle n, p\rangle \in Q\}$.

The proof that F is a homeomorphism $U \to \mathrm{MF}(Q)$ follows as in the proof of Theorem 2.3.18. □

Theorem 4.2.3. The following are equivalent over ACA_0.

1. Π^1_1-CA_0.

2. If P is a countable poset then for every G_δ subspace U of $\mathrm{MF}(P)$ there is a countable poset Q such that $U \cong \mathrm{MF}(Q)$.

3. If $\widehat{A}$ is a complete separable metric space and U is a G_δ subspace of $\widehat{A}$ then there is a complete separable metric space $\widehat{B}$ such that $U \cong \widehat{B}$.

Proof. Lemma 4.2.2 shows that (1) implies (2). To see that (2) implies (3), let $\langle \widehat{A}, d\rangle$ be a complete separable metric space and let $U = \bigcap U_i$ be a G_δ subspace of $\widehat{A}$. Working in ACA_0, we form the standard poset representation $P_{\widehat{A}}$ of $\widehat{A}$. We use the code for U to form the corresponding G_δ subspace U' of $\mathrm{MF}(P_{\widehat{A}})$ such that $U \cong U'$. We then apply (2) to obtain a poset Q such that $U' \cong \mathrm{MF}(Q)$. ACA_0 allows us to form a dense subset $\{F_i\}$ of $\mathrm{MF}(Q)$,which we transform into a dense subset $B = \{b_i\}$ of U in $\widehat{A}$. All that remains is to define a metric d' on B such that the completion of D under this metric is homeomorphic to U. We let $\langle f_i\rangle$ be a sequence of continuous functions from A to $[0, \infty)$ such that for all $z \in \widehat{A}$ and all $i \in \mathbb{N}$, $f_i(z) > 0$ if and only if $z \in U_i$. This sequence may be formed in RCA_0; see [Sim99, Lemma II.7.1]. We then let

$$d'(z, z') = d(z, z') + \sum_{i \in \mathbb{N}} 2^{-i} |f_i(z) - f_i(z')|^{-1}.$$

It can be shown that $\langle \widehat{B}, d'\rangle \cong U$.

It remains to show that (3) implies (1). Note that RCA_0 proves that each closed subset C of a complete separable metric space $\widehat{A}$ is a G_δ subspace.

For if we let $f\colon \widehat{A} \to [0,\infty)$ be such that $f(x) = 0 \Leftrightarrow x \in C$ and let $U_n = \{x \mid f(x) < 1/n\}$ then $C = \bigcap U_n$.

Brown has shown [Bro90] that Π^1_1-CA_0 is equivalent over RCA_0 to the statement "every closed subset of a complete separable metric space has a countable dense subset." We sketch a proof of Brown's reversal. Because RCA_0 proves that a Π^0_1 subset of $\mathbb{N}$ is a closed subset of the discrete metric space $\mathbb{N}$, "every closed subset of a complete separable metric space has a countable dense subset" implies ACA_0 over RCA_0. We work in ACA_0 for the rest of the reversal. Let $\langle T_i \rangle$ be a sequence of subtrees of $\mathbb{N}^{<\mathbb{N}}$. We will show that the set of $i \in \mathbb{N}$ such that T_i has a path exists. We build a tree T be putting $\langle i \rangle \frown \sigma$ into T for each $i \in \mathbb{N}$ and each $\sigma \in T_i$. Thus T is a sort of effective disjoint union of the trees $\langle T_i \rangle$. It is clear that for each $i \in \mathbb{N}$, T_i has a path if and only if T has a path whose first element is $\langle i \rangle$. Moreover, $[T]$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$. By assumption, $[T]$ has a countable dense subset. We may tell whether T_i has a path by asking whether there is a path in the countable dense subset of $[T]$ which begin with $\langle i \rangle$. This is an arithmetical question relative to the countable dense subset; thus ACA_0 suffices to define the set of those $i \in \mathbb{N}$ such that T_i has a path. This completes the reversal in Brown's theorem.

We have already shown that (2) implies that every G_δ subset of a complete separable metric space has a countable dense subset. We may apply Brown's result, because RCA_0 proves that every closed subset of a complete separable metric space is a G_δ subset of the space. The proof is thus complete. $\square$

4.3 Metrization theorems

In Section 4.3.1, we consider Urysohn's Metrization Theorem. Urysohn's Metrization Theorem for MF spaces is provable in Π^1_2-CA_0, while Urysohn's Metrization Theorem for UF spaces is provable in Π^1_1-CA_0. In Section 4.3.2, we consider Choquet's Metrization Theorem, which states that every metrizable poset space is completely metrizable. This proposition is provable in Π^1_2-CA_0 for MF spaces, and provable in Π^1_1-CA_0 for UF spaces. In Section 4.3.3, we show that the statements "Every countably based regular MF space is completely metrizable" and "Every countably based MF space is homeomorphic to a complete separable metric space" are equivalent to Π^1_2-CA_0 over Π^1_1-CA_0 (see Definitions 3.2.24 and 3.2.26). This section also establishes an important lemma which shows that coanalytic subsets of the Baire space can be represented as closed subsets of Hausdorff MF spaces.

A summary of the all the results obtained in Section 4.3 is given in Section 4.3.4.

4.3.1 Metrizability

Definition 4.3.1. *Urysohn's Metrization Theorem for UF spaces* is the proposition that every regular countably based UF space is metrizable. *Urysohn's Metrization Theorem for MF spaces* is the proposition that every regular countably based MF space is metrizable.

Theorem 4.3.2. Urysohn's Metrization Theorem for UF spaces implies ACA_0 over RCA_0.

Proof. Let f be a function from $\mathbb{N}$ to $\mathbb{N}$; we will show that the set $R = \{n \mid \exists m\,[f(m) = n]\}$ exists. We define a poset P exactly as in the proof of Theorem 4.1.2.

Note that there are only countably many unbounded filters on P. There is the filter generated by $x = \{2^i \mid i \in \mathbb{N}\}$; the upward closure of this filter will contain every element 3^j for which j is in the range of f. The other filters are in the set $\langle y_i \mid i \in \mathbb{N}\rangle$, where $y_i(n)$ is defined by induction on n as:

$$y_i(0) = 3^i,$$
$$y_i(n+1) = \begin{cases} 2^{n+1} & \text{if } 2^{n+1} \prec y_i(n), \\ y_i(n) & \text{otherwise.} \end{cases}$$

We note that we may form the sequence $\langle y_i \rangle$ in RCA_0, and that $\operatorname{ucl}(y_i) = \operatorname{ucl}(x)$ if and only if i is in the range of f.

It is immediate that UF(P) is a regular Hausdorff space, because the topology on UF(P) is discrete. Thus UF(P) is metrizable, by Urysohn's Metrization Theorem.

Let d be any metric inducing the topology on UF(P). Since x has a basic neighborhood not containing any y_i with $i \notin R$, there is an $r \in \mathbb{Q}^+$ such that $d(x, y_i) > r$ for all $i \notin R$. Conversely, if $j \in R$ then $d(x, y_j) = 0$. Recall that, by Lemma 3.2.28, if r is a real number and $x, y \in \text{UF}(P)$ the the relation $d(x,y) = r$ is Π^0_1 and the relation $d(x,y) < r$ is Σ^0_1. We have

$$j \in R \Leftrightarrow d(x, y_j) = 0 \Leftrightarrow d(x, y_j) < r$$

and we may thus form the set R by Δ^0_1 comprehension. □

Corollary 4.3.3. Urysohn's Metrization Theorem for MF spaces implies ACA_0 over RCA_0.

Proof. The poset in the previous proof satisfies $\mathrm{MF}(P) = \mathrm{UF}(P)$. □

Corollary 4.3.4. The following propositions imply ACA_0 over RCA_0. Every regular countably based MF space is weakly metrizable. Every regular countably based UF space is weakly metrizable.

Proof. The proof of Theorem 4.3.2 only requires that the space be weakly metrizable, because the only application of the hypothesis of metrizability is to show that a basic open set in the poset topology contains an open metric ball around each of its points. □

In light of Theorem 4.3.2, there is no loss of generality if we use ACA_0 as a base system, instead of RCA_0, when exploring the strength of Urysohn's Metrization Theorem.

We now define a property of poset spaces, called strong regularity, which is equivalent to regularity in ZFC. This definition is inspired by work of Matthias Schröder [Sch98] in effective topology. Schröder defines the class of "effectively regular" spaces and shows that every effectively regular space has a computable metric inducing its topology. Every poset space is an effective topological space, in the sense of Schröder, but a strongly regular poset space need not be an effectively regular space.

Definition 4.3.5. (RCA_0) Let P be a countable poset and let X be $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$. We say that X is *strongly regular* if there is a sequence $\langle R_p \mid p \in P\rangle$ of subsets of P such that $N_p = \bigcup_{q \in R_p} N_q$ for each p and $\mathrm{cl}(N_q) \subseteq N_p$ whenever $q \in R_p$.

The poset defined in the proof of Theorem 4.3.2 is strongly regular; we may in fact take $R_p = \{p\}$. This proves the following lemma.

Lemma 4.3.6. Each of the following statements implies ACA_0 over RCA_0.

1. Every strongly regular countably based MF space is metrizable.
2. Every strongly regular countably based UF space is metrizable.
3. Every strongly regular countably based MF space is weakly metrizable.
4. Every strongly regular countably based UF space is weakly metrizable.

Lemma 4.3.7. Π^1_2-CA_0 proves that every countably based regular MF space is strongly regular. Π^1_1-CA_0 proves that every countably based regular UF space is strongly regular.

Proof. Suppose that X is a regular poset space. We define R_p to be the set of all $q \preceq p$ such that $\mathrm{cl}(N_q) \subseteq N_p$. More formally, we let

$$R_p = \left\{ q \in P \;\middle|\; q \preceq p \wedge \forall x \subseteq P \left[(x \in X \wedge \neg \exists r \in x \; [r \perp q]) \Rightarrow p \in x \right] \right\}.$$

Recall that ACA_0 proves $N_r \cap N_q = \varnothing$ if and only if $r \perp q$. The formula defining R_p is Π^1_1 for UF spaces (because the predicate $x \in X$ is arithmetical) and Π^1_2 for MF spaces (because the predicate $x \in X$ is Π^1_1). Because X is regular, $N_p = \bigcup_{q \in R_p} N_q$ for all $p \in P$. □

We will show in Section 4.3.3 that Π^1_2 comprehension is required to prove that every regular countably based MF space is strongly regular.

Open Problem 4.3.8. Determine the Reverse Mathematics strength of the proposition that every regular countably based UF space is strongly regular.

Recall that the definition of metrizability requires that every open set in the poset topology is an open set in the metric topology, and *vice versa*, but does not require that there is a uniform method for converting a coded open set in the poset topology to a coded open set in the poset topology, or *vice versa*. The next two lemmas show that ACA_0 is able to uniformly convert codes for open sets in the metric topology to codes for open sets in the poset topology, and ACA_0 proves that the ability to uniformly convert codes in the other direction is equivalent to strong regularity.

Lemma 4.3.9. (ACA_0) Suppose that X is a weakly metrizable countably based poset space. There is an arithmetically defined functional Φ such that for every $S = \{\langle x_i, r_i \rangle\} \subseteq X \times \mathbb{Q}^+$ we have $\Phi(S) \subseteq P$ and $\bigcup_{\langle x_i, r_i \rangle \in S} B(x_i, r_i) = \bigcup_{p \in \Phi(S)} N_p$.

Proof. Let d be a metric on X, and let A be a countable dense subset of X. For $p \in P$, $x \in X$, and $r \in \mathbb{Q}^+$, we write $p \ll \langle x, r \rangle$ if and only if there is an $a \in A \cap N_p$ and an $s \in \mathbb{Q}^+$ such that $d(a, b) + d(a, x) + s < r$ for all $b \in A \cap N_p$. The relation $\ll$, viewed as a three-place predicate of p, x, and r, is uniformly arithmetically definable relative to d and A. For each $S = \{\langle x_i, r_i \rangle\} \subseteq X \times \mathbb{Q}^+$ let

$$\Phi(S) = \{ p \mid \exists i \, [p \ll \langle x_i, r_i \rangle] \}.$$

It is clear that Φ is uniformly arithmetically definable.

It remains to show that $\bigcup_{i \in \mathbb{N}} B(x_i, r_i) = \bigcup_{p \in \Phi(S)} N_p$ for each $S = \{\langle x_i, r_i \rangle\} \subseteq X \times \mathbb{Q}^+$. First, suppose that $p \in \Phi(S)$ and $y \in N_p$. Then for some $i \in \mathbb{N}$ we have $p \ll \langle x_i, r_i \rangle$. From the definition of $\ll$ we see that $d(y, x_i) < r$, which means $y \in B(x_i, r_i)$. For the converse, suppose that $y \in B(x_i, r_i)$ for some $i \in \mathbb{N}$. Choose $a \in A$ and $t \in \mathbb{Q}^+$ such that $d(a, y) < t$ and $d(x_i, a) + 2t < r_i$. By the definition of weak metrizability, there is a $p \in P$ such that $y \in N_p$ and $N_p \subseteq B(a, t)$. We will show that $p \ll \langle x_i, r_i \rangle$. Choose $a' \in A \cap N_p$. Then $d(x_i, a') < r_i$ so $a' \in B(x_i, r_i)$. For $b \in A \cap N_p$, $d(a', b) < t$, whence $d(x_i, a') + d(a', b) + t < r_i$. We conclude $p \ll \langle x_i, r_i \rangle$. □

Lemma 4.3.10. (ACA_0) Suppose that X is a weakly metrizable countably based poset space. The following are equivalent.

1. X is strongly regular.
2. There is an arithmetically definable functional Ψ such that for each $U \subseteq P$, $\Psi(U) \subseteq X \times \mathbb{Q}^+$ and $\bigcup_{p \in U} N_p = \bigcup_{\langle x, r \rangle \in \Phi(U)} B(x, r)$.

Proof. Let d be a metric on X, and let A be a countable dense subset of X.

First, suppose that there is a sequence $\langle R_p \mid p \in P \rangle$ witnessing the strong regularity of X. We define a relation $\ll$ on $(X \times \mathbb{Q}) \times P$ by letting $\langle x, r \rangle \ll p$ if and only if there is a $q \in R_p$ such that $B(x, r) \cap A \subseteq N_q$. It is clear that $\langle x, r \rangle \ll p$ implies $B(x, r) \subseteq N_p$, because $\operatorname{cl}(B(x, r)) \subseteq \operatorname{cl}(N_q) \subseteq R_p$. Moreover, for every $x \in N_p$ there is a $q \in R_p$ with $x \in N_q$. Thus there is a $y \in X$ and an $r \in \mathbb{Q}^+$ such that $x \in B(y, r) \subseteq N_q$. Choose $a \in A$ and $s \in \mathbb{Q}^+$ such that $d(y, a) + s < r$ and $d(x, a) < s$. Then $x \in B(a, s) \subseteq B(y, r)$. Moreover, $B(y, r) \cap A \subseteq N_q \cap A$, which implies $\langle a, r \rangle \ll p$. Thus $N_p = \bigcup \{B(a, s) \mid \langle a, s \rangle \ll p\}$ for every $p \in P$. For each $U \subseteq P$ we let

$$\Psi(U) = \{\langle a, r \rangle \mid \exists p \in U\, [\langle a, r \rangle \ll p]\}.$$

It is clear that Ψ is uniformly arithmetically definable, and it follows from the discussion above that $\bigcup_{p \in U} N_p = \bigcup_{\langle x, r \rangle \in \Psi(U)} B(x, r)$.

Next, suppose that Ψ is any arithmetically definable functional such that $\bigcup_{p \in U} N_p = \bigcup_{\langle x, r \rangle \in \Psi(U)} B(x, r)$ for each $U \subseteq P$. We wish to show that X is strongly regular. For each $p \in P$, let $V_p = \Psi(\{p\})$. Thus V_p is a sequence of elements of $X \times \mathbb{Q}^+$. Let W_p be the set of all pairs $\langle a, s \rangle \in X \times \mathbb{Q}^+$ such that there exists $\langle x, r \rangle \in V_p$ with $d(a, x) + s < r$. Note that if $\langle a, s \rangle \in T_p$ then $\operatorname{cl}(B(a, s)) \subseteq N_p$, and every $x \in N_p$ is in $B(a, s)$ for some $\langle a, s \rangle \in W_p$. Furthermore, we may uniformly define W_p from p with an arithmetical formula. Let Φ be the functional constructed in Lemma 4.3.9. For each $p \in P$ let

$$R_p = \{q \in P \mid \exists \langle a, s \rangle \in W_p\, [q \in \Phi(\{\langle a, s \rangle\})\,]\}.$$

The set R_p is uniformly arithmetically definable from p, and we may thus form the sequence $\langle R_p \rangle$ in ACA_0. It follows from the construction that $\langle R_p \rangle$ is a witness to the strong regularity of X. $\square$

We will show, in ACA_0^+, that if a countably based poset space is strongly regular then it is metrizable. The converse has been stated as Problem 4.3.8. Our proof will roughly follow the traditional proof of Urysohn's Metrization Theorem that a regular second-countable space is metrizable, which we now outline. The first step is to show that every second-countable regular space X is normal. The second step is to use the normality of X and a dependent choice principle to produce a family of continuous functions from X to $[0, \infty)$. The final step is to form a metric on X as a weighted sum of these continuous functions. The main technical difficulty in the formalization of this proof is the strong choice principles. In order to formalize the proof of Urysohn's Metrization Theorem in weak subsystems of second-order arithmetic, we combine the first two steps of the classical proof into the following lemma, which establishes a weak form of normality and encapsulates a form of dependent choice.

Lemma 4.3.11. (ACA_0^+) If X is a strongly regular countably based poset space then there is a sequence $\langle U_i \rangle$ of open sets such that:

1. There is a function $i\colon P \to \mathbb{N}$ such that $U_{i(p)} = \{p\}$ for each $p \in P$.
2. There is a function $i^\perp\colon P \to \mathbb{N}$ such that $U_{i^\perp(p)} = \{r \mid r \perp p\}$ for each $p \in P$.
3. There are functions $\nu_1, \nu_2\colon \mathbb{N} \to \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ if $\mathrm{cl}(U_i) \cap \mathrm{cl}(U_j) = \varnothing$ then $\mathrm{cl}(U_i) \subseteq U_{\nu_1(i,j)}$, $\mathrm{cl}(U_j) \subseteq U_{\nu_2(i,j)}$, and $U_{\nu_1(i,j)} \cap U_{\nu_2(i,j)} = \varnothing$.

Proof. Let $\langle R_p \rangle$ be a sequence witnessing the strong regularity of X. For any $U \subseteq P$ we define the set $U^\perp = \{q \in P \mid \forall p \in U[q \perp p]\}$. Note that $U^\perp$ may be defined by a Π^0_1 formula. For any $U \subseteq P$, we may write X as the disjoint union of $\mathrm{cl}(U)$ and $U^\perp$.

Note that there is a function $e(U, n)$ such that for every open set U we have $\{e(U, n) \mid n \in \mathbb{N}\} = \{q \mid \exists p \in U(q \in R_p)\}$. The function e is definable by an arithmetical formula with free variables U and n and a parameter for $\langle R_p \rangle$.

We define uniformly for all $U, V \subseteq P$ a sequence $\langle G_n(U, V) \mid n \in \mathbb{N} \rangle$ of open sets:

$$G_n(U, V) = \{p \in P \mid p \preceq e(U, n) \wedge \forall i \leq n\, (p \perp e(V, i))\}.$$

Note that $G_n(U,V)$ is uniformly arithmetically definable with parameters for P, $\preceq$, and $\langle R_p \rangle$. It follows from the definition of G that for any $n, m \in \mathbb{N}$ and any $U, V \subseteq P$ the sets $G_n(U,V)$ and $G_m(V,U)$ are disjoint.

We define a pair of functionals $\nu_1, \nu_2 \colon P(\mathbb{N}) \to P(\mathbb{N})$ such that $\mathrm{cl}(U) \subseteq \nu_1(U,V)$, $\mathrm{cl}(V) \subseteq \nu_2(U,V)$, and $\nu_1(U,V) \cap \nu_2(U,V) = \varnothing$ whenever U and V are open subsets of X such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \varnothing$. Let

$$\nu_1(U,V) = \{p \mid \exists n[p \in G_n(V^\perp, U^\perp)]\},$$
$$\nu_2(U,V) = \{p \mid \exists n[p \in G_n(U^\perp, V^\perp)]\},$$

and note that these sets are defined uniformly for all U, V by arithmetical formulas with the same parameters as G.

We claim that this definition of ν_1 and ν_2 satisfies the requirement stated above. Suppose $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \varnothing$. It is clear that $\nu_1(U,V)$ and $\nu_2(U,V)$ must be disjoint. Now suppose that $x \in \mathrm{cl}(U)$; then $x \in V^\perp$ and there is an n such that $x \in N_{e(V^\perp, n)}$. Moreover, $x \notin U^\perp$, which implies that for all m, $x \notin \mathrm{cl}(e(U^\perp, m))$. There is thus an $r \in P$ such that $x \in N_r$ and $r \perp e(U^\perp, i)$ for all $i \leq m$. Let p be a common extension of r and $e(V^\perp, n)$ such that $x \in N_p$; then $p \in G_n(V^\perp, U^\perp)$, whence $x \in G_n(V^\perp, U^\perp) \subseteq \nu_1(U,V)$. This shows $\mathrm{cl}(U) \subseteq \nu_1(U,V)$. Similarly, $V \subseteq \nu_2(U,V)$.

To complete the lemma, we must iterate the functions ν_1 and ν_2. It can be seen that this iteration may be carried out in ACA_0^+. The key fact is that ν_1 and ν_2 are uniformly arithmetically definable. □

The next step in our proof of Urysohn's Metrization Theorem is to construct a sequence of continuous functions from X to $[0, \infty)$.

Lemma 4.3.12. (ACA_0^+) Let X be a strongly regular countably based poset space. There is a sequence $\langle f_{p,q} \mid p \in P, q \in R_p \rangle$ of continuous functions from P to $[0, \infty)$ such that for all $p \in P$ and $q \in R_p$ we have $f_{p,q} \restriction N_q = 0$ and $f_{p,q} \restriction (X \setminus N_p) = 1$.

Proof. Let $\langle U_i \rangle$ be a sequence of open sets as in Lemma 4.3.11 (which uses ACA_0^+), with associated functions i, $i^\perp$, ν_1, and ν_2. We show how to construct $f_{p,q}$ in ACA_0 for a fixed $p \in P$ and $q \in R_p$. The description will be uniform in p and q, allowing us to construct the sequence $\langle f_{p,q} \mid p \in P, q \in R_p \rangle$ in ACA_0.

Let D denote the set of dyadic rationals in $[0,1]$:

$$D = \{a/2^n \mid 0 \leq a \leq 2^n, n \geq 0\}.$$

Each element $q \in D$ has a unique reduced form $a/2^n$ in which n is minimal. The *length* of q is $\mathrm{lh}(q) = n$, the exponent of 2 in the reduced form of q.

We begin our construction of $f = f_{p,q}$ by constructing a map g from D to $\mathbb{N}$. The map is constructed by induction on length. We let $g(0) = i(q)$ and $g(1) = i^{\perp}(p)$. This defines g for all $k \in D$ of length 1. We will use the induction hypothesis that if $k < k'$ and g is defined on k and k' then $\mathrm{cl}(U_{g(k)}) \cap \mathrm{cl}(U_{g(k')}) = \varnothing$. This hypothesis holds at the base step because $q \in R_p$.

Suppose g is defined for all k of length $< n$ and let $l = a/2^n$ have length n. We define $g(l)$ using the function ν_1 from Lemma 4.3.11:

$$g(l) = \nu_1(g((a-1)/2^n), g((a+1)/2^n)).$$

The definition of g ensures that the induction hypothesis is satisfied.

For each $k < 1 \in D$, let $V_k = U_{g(k)}$ and let $V_1 = X$. By induction, we know that $\mathrm{cl}(V_k) \subseteq V_{k'}$ whenever $k < k'$. We will construct a code for the function $f \colon P \to \mathbb{R}^+$ such that

$$f(x) = \inf\{k \in D \mid x \in V_k\}.$$

Note that $f(x) < k$ if and only if there is some $k' < k$ with $x \in V_k$.

For each $p \in P$ define

$$m(p) = \inf\{k \in D \mid N_p \cap V_k \neq 0\},$$
$$M(p) = \sup\{k \in D \mid N_p \cap V_k \neq 0\}.$$

Note that $N_p \cap V_k$ is open, and we may thus define $N_p \cap V_k = \varnothing$ with an arithmetical formula.

Claim: If $x = \langle p_i \rangle \in X$ then $\lim_{i\to\infty} m(p_i) = \lim_{i\to\infty} M(p_i) = f(x)$. It is clear that $m(p_i)$ is nondecreasing. If $m = \lim m(p_i) < f(x)$ then we may choose $k \in D$ such that $m < k < f(x)$. Now x cannot be in $\mathrm{cl}(V_k)$, since $k < f(x)$; but if $m < k$ then every neighborhood of x has nonempty intersection with V_k and thus $x \in \mathrm{cl}(V_k)$. We have reached a contradiction by assuming $m < f(x)$. Similarly, if $f(x) < M$ then there is some $k \in D$ with $f(x) < k < M$. Thus $f(x) < k$ and $x \in \mathrm{cl}(X \setminus V_k)$, which is impossible because of the definition of ν_1 and ν_2. This finishes the proof of the claim.

We build a code for a continuous function by associating each $p \in P$ with every open ball of the form $B(q, r)$ such that $q - r < m(p)$ and $q + r > M(p)$. It is not difficult to show in ACA_0 that this is a code for a continuous function and that the function encoded is f. □

Theorem 4.3.13. (ACA_0^+) Every strongly regular countably based MF space is metrizable.

Proof. Let X be a countably based poset space, let $\langle R_p \rangle$ witness the strong regularity of X, and let A be a countable dense subset of X.

The proof now follows the classical proof of Urysohn's Metrization Theorem. Let $\langle f_i \rangle$ be the sequence of functions constructed in Lemma 4.3.12. We define a coded continuous function $d\colon X \times X \to [0, \infty)$ such that

$$d(x,y) = \sum_{i \in \mathbb{N}} 2^{-i} |f_i(x) - f_i(y)|.$$

To construct the code for d, we use the fact that the terms in the series converge quickly to zero.

Claim: X is weakly metrizable with metric d. First, suppose that $x \in N_p$. Choose $q \in R_p$ with $x \in N_q$. Choose i such that $f_i \upharpoonright N_q = 0$ and $f \upharpoonright (X \setminus N_p) = 1$. Let $r = 2^{-(i+1)}$. Then $B(x,r) \subseteq N_p$, because of the way d was defined.

Conversely, fix $x \in X$ and $r \in \mathbb{Q}^+$. Let n be large enough that $r > \sum_{i=n}^{\infty} 2^{-i}$. There are only finitely many $q \in P$ such that $x \in N_q$ and there is a $p \in P$ and an $i \leq n$ such that f_i is the function that separates N_q from N_p. Choose $r \in P$ such that $x \in N_r$ and N_r is contained in N_q for all such q. Then $N_r \subseteq B(x,r)$. This completes the proof of the claim.

Because X is weakly metrizable and strongly regular, we may apply Lemmas 4.3.9 and 4.3.10 to show in ACA_0 that every open set in the metric topology is open in the poset topology every open set in the poset topology is open in the metric topology. Thus X is metrizable with metric d. □

Corollary 4.3.14. (ACA_0^+) Every strongly regular countably based UF space is metrizable.

Proof. The proof of Theorem 4.3.13 goes through in ACA_0^+ when MF spaces are replaced by UF spaces. □

Open Problem 4.3.15. Determine the Reverse Mathematics strengths of the following propositions.

1. Every strongly regular countably based MF space is metrizable.
2. Every strongly regular countably based MF space is weakly metrizable.

The strengths of the corresponding theorems for UF spaces are also unknown.

Corollary 4.3.16. Urysohn's Metrization Theorem for countably based UF spaces is provable in Π^1_1-CA_0. Urysohn's Metrization Theorem for countably based MF spaces is provable in Π^1_2-CA_0.

Proof. The proof follows from Lemma 4.3.7, Theorem 4.3.13, and Corollary 4.3.14. □

Open Problem 4.3.17. Recall that a topological space is normal if each pair of disjoint closed sets is contained in a pair of disjoint open sets. This definition, when restricted to countably based poset spaces, may be phrased as a sentence in the language of second-order arithmetic. It is well known that ZFC proves that every regular second-countable space is normal. What is the reverse mathematics strength of the proposition that every regular countably based MF space is normal? The same question may be asked for UF spaces.

4.3.2 Complete metrizability

Because poset spaces have the strong Choquet property (Lemma 2.3.29), every metrizable poset space is completely metrizable. In this section, we will show in Π^1_2-CA_0 that every regular countably based MF space is completely metrizable (see Definition 3.2.26). This will allow us to show, in Π^1_2-CA_0, that every regular countably based MF space is homeomorphic to a complete separable metric space (Definition 3.2.24). In Section 4.3.3, we show that each of these statements implies Π^1_2-CA_0 over Π^1_1-CA_0. We also show in this section that the statements "Every regular countably based UF space is completely metrizable" and "Every regular countably based UF space is homeomorphic to a complete separable metric space" are provable in Π^1_1-CA_0. We do not know the Reverse Mathematics strength of these statements for UF spaces.

Lemma 4.3.18. (RCA_0) If $\langle U_i \mid i \in \mathbb{N}\rangle$ is a sequence of open sets in a complete separable metric space $\langle \widehat{A}, d\rangle$ then there is a complete metric d' defined on $\bigcap U_i$ which gives the same topology to $\bigcap U_i$ as d. If D is countable dense subset of $\bigcap U_i$ then $\bigcap U_i$ is canonically homeomorphic to the complete separable metric space $\langle \widehat{D}, d'\rangle$.

Proof. Let $\langle f_i\rangle$ be a sequence of continuous function from $\widehat{A}$ to $[0,\infty)$ such that $f_i(x) > 0$ if and only if $x \in U_i$ for each i and every $x \in \widehat{A}$. The existence of such a sequence is provable in RCA_0; see [Sim99, Lemma II.7.1].

Define a new metric d' on $\bigcap U_i$ by the rule

$$d'(x,y) = d(x,y) + \sum_{i \in \mathbb{N}} \min \left\{ 2^{-i}, |f_i(x) - f_i(y)|^{-1} \right\}.$$

A code for the metric d' may be defined from d and $\langle f_i \rangle$ in RCA_0.

It can be shown that the topology on $\bigcap U_i$ induced by d' is the same as that induced by d, and that d' is a complete metric on $\bigcap U_i$; see [Kec95, Theorem 3.11]. □

Lemma 4.3.19. (ACA_0) Let P be a countable poset and let X be $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$. Suppose that X is metrizable. There is a complete separable metric space $\widehat{A}$ and a continuous open bijection h between X and a dense subset of $\widehat{A}$. Moreover, h is an isometry.

Proof. Assume that d is a metric on X. Let A be a countable dense subset of X. For each $p \in P$ we define $\mathrm{diam}(p) \in [0, \infty)$ as

$$\mathrm{diam}(p) = \sup\{d(a,a') \mid a, a' \in N_p \cap A\}.$$

This definition is arithmetical, and thus valid in ACA_0.

We will build a code for a continuous function h from X to the standard poset representation Q of the complete separable metric space $\langle \widehat{A}, d \rangle$. To do this, we let H be the set of all $\langle p, \langle a, r \rangle \rangle \in P \times (A \times Q)$ such that $a \in N_p$ and $\mathrm{diam}(p) < r$. Let h be the continuous function encoded by H.

Let x be any filter in X. We must show that $h(x)$ gives the correct filter in Q. Let $\langle a_{n_i} \rangle$ be a Cauchy sequence in A which converges to x. For each $m \in \mathbb{N}$ choose $p_m \in P$ such that $x \in N_p$ and $\mathrm{diam}(p) < 2^{-(m+1)}$. Choose $i \in \mathbb{N}$ large enough that $a_{n_i} \in N_p$. Then $\langle p, \langle a_{n_i}, 2^{-m} \rangle \rangle \in F$. Thus there are open balls of arbitrarily small radius associated with x.

We need to show that $H[x]$ generates a filter. Suppose that $\langle p, \langle a,r \rangle \rangle$ and $\langle q, \langle a', s \rangle \rangle$ are in $F[x]$. Then $d(x,a) < r$ and $d(x,a') < s$. Choose $\epsilon \in \mathbb{Q}^+$ such that $d(x,a)+\epsilon < r$ and $d(x,a')+\epsilon < s$. Choose $u \in P$ such that $x \in N_u$ and $N_u \subseteq B(x, \epsilon/8)$. Note that $\mathrm{diam}(u) \leq \epsilon/4$. Choose $a'' \in A \cap N_u$. Then $\langle u, \langle a'', \epsilon/4 \rangle \rangle \in F$. Now $d(a,a'') + \epsilon/4 < d(a,x) + d(x,a) + \epsilon/4 < d(a,x) + \epsilon/2 < r$, so $B(a'', \epsilon/4)$ is formally included in $B(a,r)$. Similarly $B(a'', \epsilon/4)$ is formally included in $B(a', s)$. This shows that every pair of elements in $h(x)$ have a common extension in $h(x)$.

We have thus shown that $h(x)$ is a point in Q for all $x \in \mathrm{MF}(P)$. It is clear that h is an isometry, because the restriction of h to A is an isometry. Because X is metrizable with metric d, every open subset of X in the poset topology is open in the metric topology. Because h is an isometry, we can easily convert a code for an open subset of X in the metric topology to a code for the corresponding open subset of $h(X) \subseteq \widehat{A}$. $\square$

Note that Lemma 4.3.19 does not show that the inverse of h exists.

Open Problem 4.3.20. Suppose that X is a metrizable countably based MF space. What is the Reverse Mathematics strength of the proposition that X is homeomorphic to a dense subset of a complete separable metric space? What is the strength if we require h to be an isometry? The same questions may be asked for metrizable UF spaces.

The full hypothesis of metrizability is used in the proof of Lemma 4.3.19; we do not know if this hypothesis could be replaced by weak metrizability in ACA_0. The following corollary shows that, in ACA_0, we may replace metrizability by weak metrizability if we also weaken the conclusion to say that h is weakly open.

Corollary 4.3.21. (ACA_0) Let P be a countable poset and let X be $\mathrm{UF}(P)$ or $\mathrm{MF}(P)$. Suppose that X is weakly metrizable. There is a complete separable metric space $\widehat{A}$ and a continuous weakly open bijection h between X and a dense subset of $\widehat{A}$. Moreover, h is an isometry.

Proof. The proof is analogous to that of Lemma 4.3.19. $\square$

Open Problem 4.3.22. Determine the Reverse Mathematics strength of the following statement. If X is a weakly metrizable countably based MF space then there is a continuous open bijection h from X to a dense subset of a complete separable metric space. What if we also require h to be an isometry? The same questions may be asked for weakly metrizable UF spaces.

Lemma 4.3.23. (Π^1_2-CA_0) Suppose that P is a countable poset and $X = \mathrm{MF}(P)$ is metrizable. There is an embedding f of X as a dense subset of a complete separable metric space $\widehat{A}$ and a sequence $\langle U_i \rangle$ of open sets in $\widehat{A}$ such that $f(X) = \bigcap U_i$.

Proof. The proof is inspired by the proof of [Kec95, Theorem 8.17(ii)]. That proof, however, uses the languages of games while proof here does not. Let d be a metric on X compatible with the original topology and let $A = \langle a_i \rangle$ be a dense subset of X. By Lemma 4.3.19, there is an isometric embedding f of X into $\langle \widehat{A}, d \rangle$.

We may use Π^1_2 comprehension to form the set $\{\langle a, r, p\rangle \mid B(a,r) \subseteq N_p\}$ and the set $\{\langle a, r, p\rangle \mid N_p \subseteq B(a,r)\}$. Here, $B(a,r)$ denotes a subset of X. We will use these sets as oracles for the rest of the proof.

We now construct a countable tree T consisting of certain finite sequences of the form $\langle W_0, q_0, W_1, q_1, \ldots, W_{n-1}, q_n\rangle$ or $\langle W_0, q_0, W_1, q_1, \ldots, q_n, W_n\rangle$ such that each W_i is a sequence of balls $B(a,r)$, with $a \in A$ and $r \in \mathbb{Q}^+$; $q_i \in P$ and $\operatorname{diam}(q_i) < 2^{-i}$ for $i \leq n$; and $N_{W_0} \supseteq N_{q_1} \supseteq N_{W_1} \supseteq N_{q_2} \supseteq \cdots$ and $q_0 \succeq q_1 \succeq q_2 \succeq \cdots$. We will ensure that for each n the set all W which occur in the final position of a sequence of length $2n+1$ in T forms a point-finite covering of X.

The tree is constructed by arithmetical transfinite recursion along $\mathbb{N}$. At stage n, we put sequences of length $n+1$ into the tree. At stage 0, form a point-finite covering of X; for each W in this covering put the sequence $\langle W\rangle$ into T.

At stage $2n+1$, begin by forming the set S of pairs $\langle \sigma, B\rangle$ such that $\sigma = \langle W_0, q_1, W_1, \ldots, W_n, q_n\rangle$ is in T and $B \subseteq N_{q_n}$. We may make an enumeration $\{B_{\sigma,n}\}$ such that $\langle \sigma, B\rangle \in S$ if and only if there is an n such that $B = B_{\sigma,n}$. Form a point-finite refinement of the countable collection of open sets $\{B_{\sigma,n}\}$. The refinement replaces each ball $B_{\sigma,n}$ with an open set $W_{\sigma,n}$. Put into T every sequence of the form $\sigma \frown \langle W_{\sigma,n}\rangle$.

At stage $2n+2$, for each sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n, W_{n+1}\rangle \in T$ put into T every sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n, W_{n+1}, q\rangle$ such that $q \preceq q_n$ and there exists a ball $B \in W_{n+1}$ such that $N_q \subseteq B$.

This completes the construction of T. We use T to construct a sequence $\langle U_i\rangle$ of open sets. For each n, U_n is the union of all open sets W which occur as the final open set in a sequence of length $2n+1$ in T. Each U_n may be written as the union of a set $\{B^n_i \mid i \in \mathbb{N}\}$ of basic open balls. A simple induction shows that for all $n \in \mathbb{N}$ the set U_n covers X; the key fact is that if $x \in B$ and $x \in N_p$ then there are p' with arbitrarily small diameter such that $x \in N_{p'} \subseteq B$ and $p' \preceq p$.

For any basic open ball $B = B(a,r)$ in X we let $\widehat{B}$ denote the ball $B(a,r)$ in $\widehat{A}$; thus $f(x) \in \widehat{B} \Leftrightarrow x \in f(B)$ for each $x \in X$ and each basic open ball B. For each n we form an open subset of $\widehat{U}_n$ of $\widehat{A}$ by letting $\widehat{U}_n = \{\widehat{B}^n_i \mid i \in \mathbb{N}\}$.

We show that $f(X) = \bigcap_{n\in\mathbb{N}} \widehat{U}_n$ in $\widehat{A}$. Because each U_n covers X, each $\widehat{U}_n$ covers $f(X)$; this shows that $f(X) \subseteq \bigcap_{n\in\mathbb{N}} \widehat{U}_n$. Suppose that $\hat{z}$ is a strong Cauchy sequence in $\bigcap_{n\in\mathbb{N}} \widehat{U}_n$. Consider the set of all sequences in T of the form $\langle W_0, \ldots, W_{n+1}\rangle$ such that $\hat{z} \in \widehat{W}_{n+1}$. These sequences form a subtree $T_{\hat{z}}$ of T, because if $\hat{z} \in \widehat{W}_{n+1}$ then $\hat{z} \in \widehat{W}_n$ for any sequence $\langle W_0, \ldots, W_n, q_n, W_{n+1}\rangle \in T$. The tree $T_{\hat{z}}$ is infinite, because $\hat{z} \in U_n$ for all $n \in \mathbb{N}$. $T_{\hat{z}}$ is also finitely branching, because the collection of all W appearing at level $2n$ of T is a point-finite collection of open sets. We apply König's Lemma to obtain a path $\langle W_0, q_o, W_1, q_1, \cdots \rangle$ through $T_{\hat{z}}$; it is clear that $\hat{z} \in \bigcap \widehat{W}_i$. Moreover, at most one point is in the intersection, because $\operatorname{diam}(q_i) < 2^{-i}$ for all $i \in \mathbb{N}$. The descending sequence $\langle q_i \rangle$ extends to a maximal filter $x \in X$, and this maximal filter is clearly in $\bigcap W_i$. Thus $f(x) = \hat{z}$. We have now shown $\bigcap_{n \in \mathbb{N}} \widehat{U}_n \subseteq f(X)$. □

We obtain the corresponding result for UF spaces as a corollary.

Corollary 4.3.24. (Π^1_1-CA_0) Suppose that P is a countable poset and $X = \mathrm{UF}(P)$ is metrizable. There is an embedding f of X as a dense subset of a complete separable metric space $\widehat{A}$ and a sequence $\langle U_i \rangle$ of open sets in $\widehat{A}$ such that $f(X) = \bigcap U_i$.

Proof. The proof parallels that of Lemma 4.3.23. The key difference is that, when X is a UF space, the predicates $N_p \subseteq B(a,r)$ and $B(a,r) \subseteq N_p$ are definable by Π^1_1 formulas. □

Theorem 4.3.25. (Π^1_2-CA_0) Every metrizable countably based MF space is completely metrizable and homeomorphic to a complete separable metric space.

Proof. Let X be a metrizable countably based MF space with metric d. Let A be a dense subset of X. By Lemma 4.3.19, we may embed X into the complete separable metric space $\langle \widehat{A}, d \rangle$. We use Lemma 4.3.23 to construct a sequence $\langle U_i \rangle$ of open sets of $\widehat{A}$ such that $X = \bigcap U_i$. Then Lemma 4.3.18 implies there is a metric d' on A such that $X = \langle \widehat{A}, d' \rangle$. As in the proof of Theorem 4.3.13, Π^1_2-CA_0 proves that every open set in the poset topology is an open set in the metric topology, and *vice versa*. Thus X is completely metrizable.

We show that X is homeomorphic to the standard poset representation of $\widehat{A}$ by constructing codes for both the forward map $h\colon X \to \widehat{A}$ and its inverse map.

The code for $h\colon \mathrm{MF}(P) \to \widehat{A}$ includes each condition $\langle p, a, r \rangle$ such that $a \in N_p$ and $\operatorname{diam}(N_p) < r$. If $\langle p, a, r \rangle$ is such a condition then $N_p \subseteq B(a,r)$. For otherwise there is a sequence of points $\langle a_i \rangle$ in N_p which converges to a point $x \in N_p \setminus B(a,r)$. This implies $d(a,x) = \lim d(a,a_i) \geq r$, and thus $\operatorname{diam}(N_p) \geq r$, a contradiction.

The code for the inverse map h^{-1} includes each condition $\langle a, r, p \rangle$ such that $B(a,r) \subseteq N_p$. The set of such conditions is definable by a Π^1_2 formula, so we may form the set of these conditions in Π^1_2-CA_0. We must show that h^{-1} defines a continuous bijection from $\langle \widehat{A}, d' \rangle$ to X. Let z be a Cauchy sequence on $\langle \widehat{A}, d' \rangle$. We know that there is a unique x such that $f(x) = z$. Note that for each p such that $x \in N_p$ there is a ball $B(a,r)$ such that $x \in B(a,r)$ and $B(a,r) \subseteq N_p$. Because $z \in B(a,r)$, we have $p \in h^{-1}(z)$. This shows that $x \subseteq h^{-1}(z)$. Because $x \in \mathrm{MF}(P)$, we have shown $x = h^{-1}(Z)$. Thus h^{-1} is well defined. □

Corollary 4.3.26. (Π^1_2-CA_0) Every regular countably based MF space is completely metrizable and homeomorphic to a complete separable metric space.

Proof. Let X be a regular countably based MF space. Corollary 4.3.16 shows that X is metrizable. By Theorem 4.3.25, X is completely metrizable and homeomorphic to a complete separable metric space. □

Corollary 4.3.27. (Π^1_1-CA_0) Every metrizable countably based UF space is completely metrizable and homeomorphic to a complete separable metric space.

Proof. The proof is parallel to that of Theorem 4.3.27. The necessary oracles are definable by Π^1_1 formulas when we are working with UF spaces. □

Corollary 4.3.28. (Π^1_1-CA_0) Every regular countably based UF space is homeomorphic to a complete separable metric space.

Proof. The proof is parallel to Corollary 4.3.26. □

We now show how to prove Lemma 4.3.23 in Π^1_1-CA_0 with the added assumption that the space is strongly regular. We obtain the corresponding result for UF spaces in ACA_0^+. These results allow us to reprove Corollaries 4.3.26 and 4.3.27, with the added assumption of strong regularity, in Π^1_1-CA_0 and ACA_0^+, respectively.

Lemma 4.3.29. (Π^1_1-CA_0) Suppose that X is a metrizable strongly regular countably based MF space. There is an embedding of X as a dense subset of a complete separable metric space $\widehat{A}$ and a sequence $\langle U_i \rangle$ of open sets in $\widehat{A}$ such that $f(X) = \bigcap U_i$.

Proof. The general outline of the proof is the same as for Lemma 4.3.23. Let d be a metric on X compatible with the original topology, let $A = \langle a_i \rangle$ be a dense subset of X, and let f be an embedding of X to $\widehat{A}$. Let $\langle p_i \rangle$ be an enumeration of P. Let $\langle R_p \mid p \in P \rangle$ be a sequence witnessing the strong regularity of X.

As in the proof of Lemma 4.3.23, we construct a countable tree T of sequences of the form $\langle W_0, q_0, W_1, q_1, \ldots, W_{n-1}, q_n \rangle$ or $\langle W_0, q_0, \ldots, q_n, W_n \rangle$ such that each W_i is a sequence of balls $B(a, r)$, with $a \in A$ and $r \in \mathbb{Q}^+$ and $q_i \in P$ for $i \leq n$. We will make the following requirements on the sequences in T.

1. Each W_i contains a basic open ball B such that $N_{q_{i+1}} \subseteq B$.

2. For each q_i there is a $q \in R_{q_i}$ such that $N_{W_{i+1}} \cap A \subseteq N_q \cap A$.

3. $q_{i+1} \preceq q_i$ for each $i \in \mathbb{N}$.

4. $\operatorname{diam} q_i < 2^{-i}$ for each $i \in \mathbb{N}$.

It can be seen that the collection of sequences satisfying these requirements is definable by an arithmetical formula. We note that requirement (2) implies that $N_{W_{i+1}} \subseteq N_{q_i}$ for each $i \in \mathbb{N}$.

We construct a tree by transfinite recursion along $\mathbb{N}$ as in the proof of Lemma 4.3.23. At stage 0, form a point-finite covering of X; for each W in this covering put the sequence $\langle W \rangle$ into T.

At stage $2n + 1$, begin by forming the set S of pairs $\langle \sigma, B \rangle$ such that σ is in T, $|\sigma| = n$, and $\sigma \frown \langle \{B\} \rangle$ satisfies the requirements (only requirement (2) is an issue). Define a sequence $\{B_{\sigma,n}\}$ such that $\langle \sigma, B \rangle \in S$ if and only if there is an n such that $B = B_{\sigma,n}$. Form a point-finite refinement $\{W_{\sigma,n}\}$ of $\{B_{\sigma,n}\}$ and put into T every sequence of the form $\sigma \frown \langle W_{\sigma,n} \rangle$.

At stage $2n + 2$, for each sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n, W_{n+1} \rangle \in T$ put into T every sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n, W_{n+1}, q \rangle$ which satisfies the requirements.

This completes the construction of T; note that T may be constructed in ACA_0^+ because the stages of the construction are uniformly given by arithmetic functionals. Construct a sequence $\langle U_i \rangle$ of open sets by letting U_n be the union of all open sets W which occur as the final open set in a sequence of length $2n + 1$ in T, and find an enumeration $U_n = \{B_i^n \mid i \in \mathbb{N}\}$.

We say a sequence $\sigma \in T$ *meets* a point $x \in X$ if x is in the neighborhood determined by the final element of the sequence σ; if the length of σ is odd this will be a union of metric balls, while if the length is even this will be a basic neighborhood in the poset topology. We prove that for every x and every m there is a sequence of length m in T which meets x. Let $x \in X$ be fixed; we prove the result by induction on m. The case for $m = 1$ follows immediately from the construction. The rest of the induction breaks into two cases: $m+1$ even and $m+1$ odd.

Suppose that the induction is valid through m and $m+1 = 2n+2$ is even. Then there is a sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n \rangle$ of length m in T which meets x; this means $x \in N_{q_n}$. Choose $q \in R_{q_n}$ such that $x \in N_q$. Because the metric topology is compatible with the poset topology, there must be a basic open ball B such that $x \in B$ and $B \subseteq N_q$. Thus $B \cap A \subseteq N_q \cap A$; so the sequence $\langle W_0, q_1, W_1, \ldots, W_n, q_n, \{B\} \rangle$ satisfies the requirements. Because there is at least one sequence of length $m+1$ which meets x before the point-finite refinement is made, there will be a sequence of length $m+1$ which meets x after the refinement is made.

Now suppose that the induction is valid though m and $m+1 = 2n+1$ is odd. By induction, there is a sequence $\langle W_0, q_1, W_1, \ldots, q_{n-1}, W_n \rangle$ of length m in T which meets x. Choose a basic open ball B in W_n such that $x \in B$. We can then choose $r \in P$ such that $x \in N_r$, $N_r \subseteq B$, $\operatorname{diam}(r) < 2^{-n}$, and $r \preceq q_{n-1}$. The sequence $\langle W_0, q_1, W_1, \ldots, q_{n-1}, W_n, r \rangle$ satisfies all of the requirements; this is a sequence of length $m+1$ in T which meets x.

Thus we have shown that for every $x \in X$ and every $m \in \mathbb{N}$ there is a sequence of length m in T which meets x. In particular, this implies that if we define U_n to be the union of all open sets W which occur as the final open set in a sequence of length $2n+1$ in T then U_n will cover X for each $n \in \mathbb{N}$.

The rest of the proof duplicates Lemma 4.3.23; once the open sets U_n have been constructed, the rest of the argument goes through in Π^1_1-CA_0. □

Lemma 4.3.30. (ACA_0^+) Suppose that X is a metrizable strongly regular countably based UF space. There is an embedding of X as a dense subset of a complete separable metric space $\widehat{A}$ and a sequence $\langle U_i \rangle$ of open sets in $\widehat{A}$ such that $f(X) = \bigcap U_i$.

Proof. By retracing the proof of Lemma 4.3.29, we see that the only remaining use of Π^1_1 comprehension is to show that a descending sequence extends to a maximal filter. RCA_0 proves that every filter extends to an unbounded filter. The rest of the proof of Lemma 4.3.29 uses only ACA_0^+. □

Lemma 4.3.31. (Π^1_1-CA_0) Every metrizable strongly regular countably based MF space is completely metrizable and homeomorphic to a complete separable metric space.

Proof. Let X be a metrizable MF space,with metric d, and let $\langle R_p \rangle$ witness the strong regularity of X. Let A be a countable dense subset of X. Apply

Lemma 4.3.29 and Lemma 4.3.18 to obtain a code for a complete metric d' on X.

To complete the proof that X is completely metrizable, we must to show that each open set in the poset topology is open in the metric topology induced by d', and that each open set in the metric topology induced by d' is open in the poset topology. We know that this holds for the metric topology induced by d, and we know that d and d' are compatible metrics by Lemma 4.3.18. The proof resembles the proofs of Lemmas 4.3.9 and 4.3.10 but is not quite the same. In those lemmas, we assumed that X was metrizable. Here, we are trying to prove that X is metrizable with metric d' assuming it is metrizable with metric d.

Let $U \subseteq P$ be fixed. We use arithmetical comprehension to define

$$S(U) = \{\langle a, r\rangle \in A \times \mathbb{Q}^+ \mid \exists p \in U \exists q \in R_p\, [B_{d'}(a, r) \cap A \subseteq N_q \cap A]\}$$

We claim that $\bigcup\{N_p \mid p \in U\} = \bigcup\{B_{d'}(a, r) \mid \langle a, r\rangle \in S(U)\}$. It is enough to prove that $N_p = \bigcup\{B_{d'}(a, r) \mid \exists q \in R_p\, [B_{d'}(a, r) \cap A \subseteq N_q \cap A]\}$. If $x \in B_{d'}(a, r)$ and $B_{d'}(a, r) \cap A \subseteq N_q$ then $x \in \mathrm{cl}(N_q) \subseteq N_p$. Conversely, if $x \in N_p$ then there is a $q \in R_p$ with $x \in N_q$. We know that there is a ball $B_d(a, r)$ with $x \in B_d(a, r)$ and $B_d(a, r) \subseteq N_q$. Thus there is a ball $B_{d'}(a', r')$ such that $x \in B_{d'}(a', r')$ and $B_{d'}(a', r') \subseteq B_d(a, r) \subseteq N_q$. This finishes the proof that an open set in the poset topology in open in the topology induced by d'.

Now suppose that $x \in X$ and $r \in \mathbb{Q}^+$ are given. We will uniformly construct a set $U \subseteq P$ such that $B_{d'}(x, r) = \bigcup_{p \in U} N_p$. Because the proof is uniform, we may write each open set in the metric topology induced by d' as an open set in the poset topology.

Let U be the set of all $p \in P$ such that there is an $a \in A \cap N_p$ and an $s \in \mathbb{Q}^+$ such that $d'(a, b) + d'(a, x) + s < r$ for all $b \in A \cap N_p$. It follows that if $p \in U$ and $y \in N_p$ then $y \in B_{d'}(x, r)$. Conversely, if $y \in B_{d'}(x, r)$ then there is an $a \in A$ and an $s \in \mathbb{Q}^+$ such that $d'(x, a) + 2s < r$ and $d'(a, y) < s$. This follows from the fact that A is a dense subset of X in all three topologies at hand. Since $y \in B_{d'}(a, s)$, and the metric d' is compatible with d, there is a metric ball $B_d(b, t) \subseteq B_{d'}(a, s)$ with $y \in B_d(b, t)$. Since X is metrizable with metric d, there is a $p \in P$ such that $x \in N_{q'} \subseteq B_d(b, t)$. Because $B_d(b, t) \subseteq B_{d'}(a, s)$, we see that $d'(a, b) + d'(a, x) + s < r$ for all $b \in A \cap N_p$. We have now shown that $B_{d'}(x, r) = \bigcup_{p \in U} N_p$.

We have now shown that X is completely metrizable with metric d. It remains to show that X is homeomorphic to a complete separable metric space. The previous claim shows that we may use arithmetical comprehension to uniformly convert codes for open subsets in the poset topology on

X to codes for open sets in the metric topology on X, and *vice versa*. It is thus straightforward to construct codes for a continuous open bijection $h\colon X \to \langle \widehat{A}, d' \rangle$ and its inverse. □

Lemma 4.3.32. (ACA_0^+) Each strongly regular countably based UF space is completely metrizable and homeomorphic to a complete separable metric space.

Proof. The proof is parallel to the proof of Lemma 4.3.31. □

4.3.3 Metrization and Π^1_2-CA_0

In this section, we show that the following propositions are equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

1. Every regular countably based MF space is strongly regular.
2. Every regular countably based MF space is completely metrizable.
3. Every regular countably based MF space is homeomorphic to a complete separable metric space.

(See Definitions 3.2.26 and 3.2.24.) These reversals are remarkable because no other theorem of core mathematics provable in second-order arithmetic is known to imply Π^1_2 comprehension. These results give evidence for the popular opinion that general topology is somehow less constructive than other mathematics.

We remark that the base system for there reversals is Π^1_1-CA_0, which is quite strong. In order to have a good theory of countably based MF spaces, it is necessary to assume Π^1_1 comprehension in order to show that every filter on a countable poset extends to a maximal filter (see Theorem 4.1.5). This extension property, a consequence of Zorn's Lemma, is a basic aspect of maximal filters. Thus, the use of Π^1_1-CA_0 as a base system is acceptable when we are dealing with MF spaces.

Theorem 4.3.33. The proposition that every countably based regular MF space is homeomorphic to a complete separable metric space is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

We have already shown that Π^1_2-CA_0 proves that every regular countably based MF space is homeomorphic to a complete separable metric space (Corollary 4.3.26). We postpone the rest of the proof of Theorem 4.3.33 to prove two lemmas. The first lemma gives a sufficient condition for Π^1_2-CA_0 to hold.

Lemma 4.3.34. (Π^1_1-CA_0) There is a Π^1_1 formula $\Psi(n, f, h)$ with one free number variable and two free set variables such that for each $n \in \mathbb{N}$ and $h \in \mathbb{N}^\mathbb{N}$ there is at most one $f \in \mathbb{N}^\mathbb{N}$ such that $\Phi(n, f, h)$ holds, and the existence of the set $\{n \mid \exists f \Psi(n, f, h)\}$ for every h implies Π^1_2-CA_0.

Proof. Let $\Phi(n, h)$ be a Σ^1_2 formula such that for each Σ^1_2 formula $\Theta(n, h)$ with the free variables shown there is an m such that $\forall h \forall n(\Theta(n, h) \Leftrightarrow \Phi(2^m 3^n, h))$ holds. The existence of such a formula is provable in ACA_0; see [Sim99, Lemma V.1.4].

Let $\Phi(n, h) \equiv \exists f\, \Theta(n, f, h)$, where Θ is Π^1_1. We apply Π^1_1 uniformization (which is provable in Π^1_1-CA_0; see [Sim99, Lemma VI.2.1]) to Θ to obtain a formula $\Psi(n, f, h)$ such that

$$\forall n\, \forall f\, \forall g\, [(\Psi(n, f, h) \wedge \Psi(n, g, h)) \Rightarrow f = g].$$

and

$$\forall n\, [\, \exists f\, \Theta(n, f, h) \Leftrightarrow \exists g\, \Psi(n, g, h)].$$

It is immediate that Ψ satisfies the conclusions of the lemma. □

The next lemma shows that every coanalytic subspace of Baire space may be represented as a closed subset of a Hausdorff MF space.

Lemma 4.3.35. (Π^1_1-CA_0) Let S be a coanalytic subset of $\mathbb{N}^\mathbb{N}$. There is a countable poset P such that $\mathrm{MF}(P)$ is Hausdorff and there is a closed subset C of $\mathrm{MF}(P)$ such that C is homeomorphic to S, which inherits a topology as a subspace of $\mathbb{N}^\mathbb{N}$. Moreover, $\mathrm{MF}(P)$ is a regular space if S is finite.

Proof. Let $\Psi(X)$ be a Π^1_1 formula with one free set variable such that $S = \{X \mid \Psi(X)\}$. There may be number and set parameters in Ψ. Write Ψ in normal form: there is a Δ^0_0 formula ρ such that $\Psi(X) \Leftrightarrow \forall Y\, \exists n\, \rho(X[n], Y[n])$ for all $X \in \mathbb{N}^\mathbb{N}$.

Let

$$P = \{\langle \sigma \rangle \mid \sigma \in \mathbb{N}^{<\mathbb{N}}\} \cup \{\langle \sigma, \tau \rangle \mid \sigma, \tau \in \mathbb{N}^{<\mathbb{N}} \wedge |\sigma| = |\tau| \wedge \rho(\sigma, \tau)\}.$$

The order on P is smallest order containing the following:

1. $\langle \sigma \rangle \prec \langle \sigma' \rangle$ for all $\sigma' \subseteq \sigma$
2. $\langle \sigma, \tau \rangle \prec \langle \sigma', \tau' \rangle$ for all $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$
3. $\langle \sigma, \tau \rangle \prec \langle \sigma' \rangle$ for all $\sigma' \subseteq \sigma$.

We will now characterize all the filters in $\mathrm{MF}(P)$ into three disjoint classes. *Class 1* consists of the principal filters on P. *Class 2* consists of the nonprincipal filters which contain an element of P of the form $\langle\sigma,\tau\rangle$. *Class 3* consists of the nonprincipal filters which do not contain an element of the form $\langle\sigma,\tau\rangle$. It is clear that this list is exhaustive. The filters in class 1 are generated by minimal elements of P, while all other filters contain an infinite strictly descending sequence. If F is a filter in class 2 then there must be $X_F, Y_F \in \mathbb{N}^{\mathbb{N}}$ such that $F = \{\langle X_F[m], Y_F[m]\rangle \mid m \in \mathbb{N}\}$; clearly $\forall m\, \rho(X_F[m], Y_F[m])$ holds and thus $\Psi(X_F)$ is false. A filter G in class 3 has no minimal element but does not contain a condition of the form $\langle\sigma,\tau\rangle$. Thus every condition in G is of the form $\langle\sigma\rangle$ and there is an $X_G \in \mathbb{N}^{\mathbb{N}}$ such that $G = \{\langle X_G[m]\rangle \mid m \in \mathbb{N}\}$. Because G is maximal, there must not be a $Y \in \mathbb{N}^{\mathbb{N}}$ such that $\forall m\, \rho(X_G[m], Y[m])$ holds. Thus $\Psi(X_G)$ holds. This shows that the filters in class 3 are in correspondence with the elements of $\{X \mid \Psi(X)\}$. The subspace topology of $\mathrm{MF}(P)$ on the set of maximal filters of class 3 is clearly the same as the Baire topology.

The set of all the filters of class 3 is closed in $\mathrm{MF}(P)$, because it is the complement of the open set $\{\langle\sigma,\tau\rangle \in P\}$.

The proof that $\mathrm{MF}(P)$ is Hausdorff requires several cases. We prove the only nontrivial case. Suppose that $F \in \mathrm{MF}(P)$ is a filter of class 2 and $G \in \mathrm{MF}(P)$ is of class 3. As above, let $F = \{\langle X_F[m], Y_F[m]\rangle\}$ and let $G = \{\langle X_G[m]\rangle\}$. Since $F \neq G$, $X_F \neq X_G$. Thus for some m we have $X_G[m] \perp X_F[m]$. Thus $G \in N_{\langle X_G[m]\rangle}$, $F \in N_{\langle X_F[m]\rangle}$, and $N_{X_G[m]} \cap N_{X_F[m]} = \varnothing$.

Now assume that $\{X \mid \Psi(X)\}$ is finite. We show that $\mathrm{MF}(P)$ is regular. Because $\mathrm{MF}(P)$ satisfies the T_1 axiom, we only need to show that $\mathrm{MF}(P)$ satisfies the T_3 axiom. That is, given a filter $F \in \mathrm{MF}(P)$ and a neighborhood N_p of F we must show that there is a neighborhood N_q of F such that $\mathrm{cl}(N_q) \subseteq N_p$. The proof divides into three cases, depending on the class of filter that F belongs to.

If F is in class 1, then we may take q to be the minimal element of P which generates F; for $N_q = \{F\}$ and thus $N_q = \mathrm{cl}(N_q)$.

If F is in class 2, and there is no filter of class 3 in N_p, then N_p is closed. If there is a filter G of class 3 in N_p, then it may happen that $G \in \mathrm{cl}(N_p)$. But because G is in class 3, it must be that there is a condition $q = \langle\sigma,\tau\rangle \in F$, with $q < p$, and a condition $\langle\sigma'\rangle \in G$ such that $\sigma \perp \sigma'$; otherwise $G \subseteq F$, which is impossible. By extending σ, σ' if necessary, we can ensure that there is no $G' \in C$ at all with $\langle\sigma\rangle \in G'$. This is because C is finite. Thus $C \cap \mathrm{cl}(N_q) = \varnothing$. Thus any filter in the closure of N_q is of class 1 or class 2. It is clear that a filter of class 1 or 2 in the closure of N_q

is actually in N_q; thus $F \in \text{cl}(N_q) \subseteq N_p$.

If F is in class 3, then p is of the form $\langle\sigma\rangle$. We claim that N_p is closed. Let G be a filter not in N_p. If G is of class 1 then G is isolated and thus $G \notin \text{cl}(N_p)$. If G is in class 2 then G must be in a neighborhood of the form $\langle\sigma'\rangle$ with $\sigma \perp \sigma'$; thus $G \notin \text{cl}(N_p)$. If G is of class 3 then there is a σ' such that $G' \in N_{\langle\sigma'\rangle}$ and $\sigma \perp \sigma'$; thus $G' \notin \text{cl}(N_p)$. This shows that N_p is closed.

We have exhausted the three cases; thus $\text{MF}(P)$ is regular. □

Proof of Theorem 4.3.33. Let $\Psi(n, f, h)$ be the Π^1_1 formula constructed in Lemma 4.3.34 and let $h \in \mathbb{N}^\mathbb{N}$ be fixed. For each $n \in \mathbb{N}$ we construct a poset P_n by applying Lemma 4.3.35 to the Π^1_1 formula $\Psi(f)$, where n and h are held constant.

Let P be the disjoint union of the posets $\{P_n\}$, such that $p \perp q$ if $p \in P_n$ and $q \in P_m$ for $n \neq m$. The topological space $\text{MF}(P)$ is the topological disjoint union of the spaces $\text{MF}(P_n)$. Thus $\text{MF}(P)$ is regular, because $\text{MF}(P_n)$ is regular for each $n \in \mathbb{N}$.

By assumption, there is a homeomorphism ϕ from $\text{MF}(P)$ to a complete separable metric space $\langle\widehat{X}, d\rangle$. We may use the homeomorphism to find a closed set $C' \subseteq \widehat{X}$ such that $C \cong C'$. Working in Π^1_1-CA_0, we may form a countable dense subset $\langle c'_i\rangle$ of C'. Because every point of C' is isolated in C', we see that this dense subset must actually equal C'. Now for all n we have $\exists f \Psi(n, f)$ if and only if $\exists m \Psi(n, \phi^{-1}(c'_m))$; so we may form the set $\{n \mid \exists f \Psi(n, f, h)\}$ using Π^1_1 comprehension. Because h was arbitrary, this set exists for each h; this implies Π^1_2-CA_0. □

The proof of Theorem 4.3.33 would still be valid if we chose a different method for coding homeomorphisms, so long as we are able to uniformly define the images and preimages of points and open sets with arithmetical formulas.

Corollary 4.3.36. The proposition that every countably based regular MF space is completely metrizable is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

Proof. Construct the poset P and closed subset C of $\text{MF}(P)$ exactly as in the proof of Theorem 4.3.33. Assume that d is a complete metric on $\text{MF}(P)$. Because C is a closed subset of $\text{MF}(P)$ in the poset topology, C is a closed subset in the metric topology. Thus there a sequence U of metric balls on $\text{MF}(P)$ such that $C = \text{MF}(P) \setminus U$. For each $n \in \mathbb{N}$ let A_n be a countable dense subset of $\text{MF}(P_n)$.

For each $n \in \mathbb{N}$, the poset P_n has a filter of class 3 if and only if there is a Cauchy sequence of filters in A_n which converges to a point not in U.

This can be seen to be a Σ^1_1 property of n; working in Π^1_1-CA_0, we may form the set of such n. This completes the proof. □

Corollary 4.3.37. The proposition that every countably based regular MF space is strongly regular is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

Proof. A proof in Π^1_2-CA_0 that every countably based regular poset space is strongly regular is given in Lemma 4.3.7. To prove the reversal, we reason in Π^1_1-CA_0. Let X be a countably based regular MF space; by assumption, X is strongly regular. Thus X is homeomorphic to a complete separable metric space (Corollary 4.3.26). This shows that every countably based regular MF space is homeomorphic to a complete separable metric space. We apply Theorem 4.3.33 to show that Π^1_2-CA_0 holds. □

The Reverse Mathematics strength of the statement "every countably based regular UF space is strongly regular" is unknown (see Problem 4.3.8). The next example is closely related to Example 2.3.7.

Example 4.3.38. We construct a computable poset P such that MF(P) is Hausdorff but not regular. Let C be the set of functions in $\mathbb{N}^\mathbb{N}$ which are eventually zero. Because C is Π^1_1 (actually, C is Σ^0_2), we may apply Lemma 4.3.35 to obtain a countable poset P such that C is homeomorphic to a closed subspace of MF(P). The lemma shows that MF(P) is Hausdorff. It can be seen that MF(P) is not regular, using the fact that C is dense in $\mathbb{N}^\mathbb{N}$.

4.3.4 Summary

We now summarize the results we have obtained regarding metrization of poset spaces.

The following statements about MF spaces hold.

1. "Every countably based regular MF space is metrizable" is provable in Π^1_2-CA_0 and implies ACA_0 over RCA_0 (Corollary 4.3.16 and Corollary 4.3.3).

2. "Every countably based regular MF space is completely metrizable" is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0 (Corollary 4.3.36).

3. "Every countably based regular MF space is homeomorphic to a complete separable metric space" is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0 (Theorem 4.3.33).

4. "Every countably based regular MF space is strongly regular" is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0 (Corollary 4.3.37).

5. "Every strongly regular countably based MF space is metrizable" is provable in ACA_0^+ and implies ACA_0 over RCA_0 (Corollary 4.3.13 and Lemma 4.3.6).

6. "Every metrizable strongly regular countably based MF space is completely metrizable" is provable in Π^1_1-CA_0 (Lemma 4.3.31).

7. "Every metrizable strongly regular countably based MF space is homeomorphic to a complete separable metric space" is provable in Π^1_1-CA_0 (Lemma 4.3.31).

It is an open problem to determine the exact strength Urysohn's Metrization Theorem for MF spaces, which is statement (1) above. The strengths of statements (5) and (6) are also unknown.

The following statements about UF spaces hold.

1. "Every countably based regular UF space is metrizable" implies ACA_0 over RCA_0 and is provable in Π^1_1-CA_0 (Corollary 4.3.16 and Theorem 4.3.2).

2. "Every countably based regular UF space is completely metrizable" implies ACA_0 over RCA_0 and is provable in Π^1_1-CA_0 (Corollary 4.3.28 and Theorem 4.3.2).

3. "Every countably based regular UF space is homeomorphic to a complete separable metric space" implies ACA_0 over RCA_0 and is provable in Π^1_1-CA_0 (Theorem 4.3.27 and Theorem 4.3.2).

4. "Every countably based regular UF space is strongly regular" is provable in Π^1_1-CA_0 (Lemma 4.3.7)

5. "Every strongly regular countably based UF space is metrizable" is provable in ACA_0^+ and implies ACA_0 over RCA_0 (Corollary 4.3.14 and Lemma 4.3.6).

6. "Every strongly regular countably based UF space is completely metrizable" is provable in ACA_0^+ (Lemma 4.3.32).

7. "Every strongly regular countably based UF space is homeomorphic to a complete separable metric space" is provable in ACA_0^+ (Lemma 4.3.32).

It is an open problem to determine the exact strengths of the statements in (1)–(6).

Remark 4.3.39 (Weak metrizability). It is possible to reprove the results in Section 4.3, replacing metrizability with weak metrizability (Definition 3.2.26). Note that metrizability implies weak metrizability over RCA_0, and thus the other results we have obtained about metrizability give lower bounds on the strength of the corresponding results for weak metrizability. The following results about weak metrizability are known.

1. "Every regular countably based MF space is weakly metrizable" is provable in Π^1_2-CA_0 and implies ACA_0 over RCA_0 (Corollary 4.3.4 and Corollary 4.3.16).
2. "Every strongly regular countably based MF space is weakly metrizable" is provable in ACA_0^+ and implies ACA_0 over RCA_0 (Corollary 4.3.4 and Corollary 4.3.13).
3. "Every regular countably based UF space is weakly metrizable" is provable in Π^1_1-CA_0 and implies ACA_0 over RCA_0 (Corollary 4.3.4 and Corollary 4.3.16).
4. "Every strongly regular countably based UF space is weakly metrizable" is provable in ACA_0^+ and implies ACA_0 over RCA_0 (Corollary 4.3.4 and Corollary 4.3.14).

We do not know the precise Reverse Mathematics strength of any theorem involving weak metrizability.

4.4 Compactness and sequential compactness

In this section, we consider the Reverse Mathematics of compact poset spaces. We show, in RCA_0, that the sequential closure of a countable set coincides with the full closure of the set. We prove that ACA_0 is equivalent over WKL_0 to the proposition that every sequence in a compact poset space has a convergent subsequence. The remainder of the section gives a formalization of one-point compactifications in second-order arithmetic.

Definition 4.4.1. (RCA_0) A poset space is *compact* if every covering of the space by open sets has a finite subcovering. More formally, X is compact if

$$\forall \langle U_i \rangle \left[\forall x \in X \, \exists n \; (x \in U_n) \Rightarrow \exists m \, \forall x \in X \, \exists n < m \; (x \in U_n) \right].$$

Proposition 4.4.2. RCA_0 proves that every closed subspace of a countably based compact poset space is compact.

A point x is in the closure of a set C if every neighborhood of x contains an element of C; x is in the sequential closure of C if there is a sequence of elements of C which is eventually inside every neighborhood of x. The next lemma is a formalized version of the well-known theorem that the sequential closure of a subset of a second-countable space is the same as the closure.

Lemma 4.4.3. (RCA_0). Let X be a countably based poset space. Suppose that $C = \langle x_n \rangle$ is a sequence of points in X. If a point $x \in X \setminus C$ is in the closure of C then there is a subsequence $\langle x_{n_i} \rangle$ converging to C.

Proof. Recall that a point $x \in X$ is coded as a descending sequence of elements of P, and thus the relation $x \in N_p$ is Σ^0_1.

Let $x = \langle p_i \rangle$ be a point in the closure of C. We construct a sequence $\langle n_i \rangle \subseteq \mathbb{N}$ by induction. Let $n_0 = 0$. Given n_i, consider the set

$$M_i = \{m \mid (m)_1 > n_i \wedge p_{(m)_1} \in x \wedge p_{(m)_1} \in x_{(m)_0}\},$$

which is nonempty because $x \in \mathrm{cl}(C)$. Let $n_{i+1} = (m_i)_0$, where m_i is the least element of M_i. The sequence $\langle n_i \rangle$ thus defined may be defined in RCA_0. It is immediate that $\langle x_{n_i} \rangle$ converges to x. □

Let $C = \langle x_n \rangle$ be a countable subset of a poset space. A subsequence $\langle x_{n_i} \rangle$ is *convergent* if there exists a point x such that the subsequence is eventually inside every neighborhood of x.

Lemma 4.4.4. (ACA_0) Every sequence of points in a compact countably based poset space has a convergent subsequence.

Proof. Let $\langle x_i \rangle$ be a sequence of points in X with no convergent subsequence. Then for any n, the set $C_n = \langle x_i \mid n \leq i \rangle$ is closed; this follows from Lemma 4.4.3. Moreover, no point can occur infinitely often in the sequence $\langle x_i \rangle$. We build a sequence of open sets $\langle U_i \rangle$ by letting $U_i = \mathrm{UF}(P) \setminus C_n$. Note that $X = \bigcup U_i$, but no finite subcovering of U_i covers X. Thus X is not compact. □

Theorem 4.4.5. The following are equivalent over WKL_0:

1. ACA_0
2. Every sequence of points in a compact countably based poset space has a convergent subsequence.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 4.4.4. We prove the converse implication. We will show that (2) implies that every sequence of rationals in the unit interval $[0,1]$ has a convergent subsequence. This in turn implies ACA_0 over RCA_0; see [Sim99, Theorem III.2.7].

Let $P = \{\langle q, r\rangle \mid q \in \mathbb{Q} \cap [0,1] \wedge r \in \mathbb{Q}^+\}$ and order P by letting $\langle q, r\rangle \preceq \langle q', r'\rangle$ if $|q - q'| + r < r'$. RCA_0 proves that $\mathrm{UF}(P)$ is homeomorphic to the complete separable metric space $[0,1]$ with its usual topology. Thus WKL_0 proves that P is compact, as WKL_0 proves that $[0,1]$ has the Heine-Borel property (see [Sim99, Theorem IV.1.5]).

Given a sequence $\langle r_i\rangle$ of rationals in $\mathbb{Q} \cap [0,1]$, we construct a sequence $S = \langle x_i\rangle$ of points in $\mathrm{UF}(P)$ by using the homeomorphism. Statement (2) implies that S has a convergent subsequence. Thus $\langle r_i\rangle$ has a convergent subsequence in $[0,1]$. $\square$

Open Problem 4.4.6. Determine the Reverse Mathematics strength of the Tychonoff theorem for countably based MF spaces, which states that every countable product of compact countably based poset spaces is compact.

The *one-point compactification* of a locally compact Hausdorff space X is obtained by adding a new point, x_∞, to X. The topology on $X \cup \{x_\infty\}$ is generated by the basis consisting of the open sets on X together with each set $(X \cup \{x_\infty\}) \setminus \mathrm{cl}(U)$, where $U \subseteq X$ is an open set with compact closure. It is known that the one-point compactification of a locally compact Hausdorff space is a compact Hausdorff space, and is thus metrizable if X is second countable; see [Kel55, Chapter 5].

The next proposition shows that our definition of the one-point compactification is the same as the classical definition (as given, for example, in Kelley [Kel55, Chapter 5]).

Proposition 4.4.7. (ZFC) Let X be a locally compact Hausdorff space and let U be a subset of the one-point compactification of X such that $x_\infty \in U$. Then U is open if and only if $X \setminus U$ is a compact subset of X.

Proof. Suppose that U is open in the one-point compactification of X. Then $X \cap U$ is open in the topology on X, and thus $X \setminus U$ is closed in the topology on X. Because U contains a basic open neighborhood of x_∞, the complement $X \setminus U$ is contained in a compact subset of X. Thus $X \setminus U$ is compact.

Conversely, let C be a compact subset of X. We show that $U = (X \setminus C) \cup \{x_\infty\}$ is open in the one-point compactification of U. Any point of $X \setminus C$ has an open neighborhood disjoint from C. We only need to show that x_∞ has an open neighborhood disjoint from C. Let V be the set of all precompact open sets which have nonempty intersection with C; this is a covering of C because X is locally compact. Let V' be a finite subset of V such that $C \subseteq \bigcup V'$. Now $\bigcup V'$ is open and precompact, because it is a finite union of precompact open sets. Thus $X \setminus \mathrm{cl}(V')$ is a basic open neighborhood of x_∞ in the one-point compactification of X, and this open neighborhood is disjoint from C. □

We say that a countably based MF space X is *strongly locally compact* if there is a sequence $\langle U_i \rangle$ of open subsets of X such that $\mathrm{cl}(U_i)$ is compact for each $i \in \mathbb{N}$ and every $x \in X$ is in U_j for some $j \in \mathbb{N}$.

Open Problem 4.4.8. Determine the Reverse Mathematics strength of the following proposition, which is provable in second-order arithmetic. If X is a locally compact Hausdorff countably based MF space then X is strongly locally compact. The same question may be asked for UF spaces.

Theorem 4.4.9. (ACA_0) Let P be a countable poset such that $\mathrm{MF}(P)$ is a strongly locally compact Hausdorff space. There is a countable poset Q such that $\mathrm{MF}(Q)$ is the one-point compactification of $\mathrm{MF}(P)$. This means that $\mathrm{MF}(Q)$ is a compact Hausdorff space and there is a point $x_\infty \in \mathrm{MF}(Q)$ such that $\mathrm{MF}(P)$ is homeomorphic to $\mathrm{MF}(Q) \setminus \{x_\infty\}$.

Proof. For $p \in P$ and $U \subseteq P$, we write $p \preceq U$ to mean that $p \preceq q$ for some $q \in U$. The poset Q consists of those elements of p such that $p \preceq U_k$ for some k along with countably many new elements $\{q_i \mid i \in \mathbb{N}\}$. For each $p \in P \cap Q$, we put $p \preceq_Q q_j$ if $p \perp U_k$ for $k \leq j$. The order on Q is the transitive closure of these relations and the order on $P \cap Q$.

The filter $x_\infty = \{q_i \mid i \in \mathbb{N}\}$ is a maximal filter on Q; given any $p \in P \cap Q$, there is a U_j such that $p \preceq U_j$, and thus $p \perp q_j$. Conversely, if $x \in \mathrm{MF}(Q)$ and there is a $p \in P$ such that $p \in x$, we claim that $x' = \mathrm{ucl}(x \cap P)$ is a maximal filter on P. Suppose not; then there is a $p' \in P$ such that $x \cup \{p'\}$ extends to a filter on P. This extension filter contains a common extension r of p and p'; thus $r \preceq U_j$ whenever $p \preceq U_j$. Because $x \in \mathrm{MF}(Q)$, we see that $r \in x$, which implies $p' \in \mathrm{ucl}(x \cap P)$.

We have thus shown that $\mathrm{MF}(Q)$ is in one-to-one correspondence with $\mathrm{MF}(P) \cup \{x_\infty\}$. Moreover, the set $\{\langle 0, p, p \rangle \mid p \in P \cap Q\}$ encodes a homeomorphism from $\mathrm{MF}(P)$ to $\mathrm{MF}(Q) \setminus \{x_\infty\}$.

It remains to show that $\mathrm{MF}(Q)$ is a compact Hausdorff space; the proof in ACA_0 mirrors the classical proof.

To show that $\mathrm{MF}(Q)$ is Hausdorff, fix distinct $x, y \in \mathrm{MF}(Q)$. If neither of x, y is x_∞ then there are disjoint neighborhoods of x, y in Q because there were disjoint neighborhoods in P. If, say $x = x_\infty$, then there is some j such that $y \in U_j$. Thus N_{q_j} and U_j are disjoint neighborhoods of x_∞ and y, respectively.

To see that $\mathrm{MF}(Q)$ is compact, let $\{V_i\}$ be an arbitrary covering of $\mathrm{MF}(Q)$ by open sets. Then there is a j such that $x_\infty \in V_j$, and thus there is a k such that $N_{q_k} \subseteq V_j$. Because $\mathrm{cl}(U_k)$ is compact and $\{V_i \mid i \in \mathbb{N}\}$ covers $\mathrm{cl}(U_k)$, there is an l such that $\{V_i \mid i \leq l\}$ covers U_k. Thus $\{V_i \mid i \leq l\} \cup \{V_j\}$ covers $\mathrm{MF}(Q)$. □

Corollary 4.4.10. (ACA_0) Let P be a countable poset such that $\mathrm{UF}(P)$ is a strongly locally compact Hausdorff space. There is a countable poset Q such that $\mathrm{UF}(Q)$ is the one-point compactification of $\mathrm{UF}(P)$, that is, $\mathrm{UF}(Q)$ is a compact Hausdorff space and there is a point $x_\infty \in \mathrm{UF}(Q)$ such that $\mathrm{UF}(P)$ is homeomorphic to $\mathrm{UF}(Q) \setminus \{x_\infty\}$.

Proof. Define a poset Q exactly as in the proof of Theorem 4.4.9. The filter $x_\infty = \{q_i \mid i \in \mathbb{N}\}$ is a unbounded in Q. Any other unbounded filter on Q contains an element of $Q \cap P$, and thus must be unbounded in P. Thus $\mathrm{UF}(Q)$ is in one-to-one correspondence with $\mathrm{UF}(P) \cup \{x_\infty\}$. The rest of the proof of Theorem 4.4.9 goes through unchanged. □

Lemma 4.4.11. (ACA_0) Every locally compact Hausdorff countably based poset space is regular.

Proof. We formalize the classical proof, as described in [Kel55, Chapter 5]. Let P be a countable poset such that $\mathrm{MF}(P)$ is a compact Hausdorff space. We wish to show that $\mathrm{MF}(P)$ is regular. Let $x \in \mathrm{MF}(P)$ and $p \in P$ be such that $x \in N_p$. Because X is locally compact, we may assume that p is precompact.

Form the set $U = \{q \in P \mid \exists r \in P[r \perp q \wedge x \in N_r]\}$. Then every $y \in \mathrm{MF}(P)$ distinct from x is in N_U, because $\mathrm{MF}(P)$ is Hausdorff. Thus $U \cup \{p\}$ gives an open covering of X. In particular, $U \cup \{p\}$ is an open covering of $\mathrm{cl}(N_p)$. Let $V = \{p_0, \ldots, p_n, p\}$ be a finite subset of $U \cup \{p\}$ such that $\mathrm{cl}(N_p) \subseteq N_V$. It must be that $p \in V$, because no other basic open set in U contains x. Moreover, $x \notin N_{p_i}$ for $i \leq n$. Let $W_i = \{q \in P \mid q \perp p_i\}$ for each $0 \leq i \leq n$. Then $x \in W_i$ for $i \leq n$ and thus there is an $r \in x$ such that $r \preceq p$ and $r \preceq W_i$ for $i \leq n$. It is straightforward to show that $\mathrm{cl}(N_r) \subseteq N_p$. We have thus shown that $\mathrm{MF}(P)$ is regular. □

Corollary 4.4.12. (Π^1_2-CA_0) If a Hausdorff countably based MF space is strongly locally compact then it is metrizable, completely metrizable, and homeomorphic to a complete separable metric space.

Proof. Let P be a countable poset such that $X = \text{MF}(P)$ is strongly locally compact. We reason in Π^1_2-CA_0. By Corollary 4.4.10, X embeds in its one-point compactification Y. By Lemma 4.4.11, Y is regular. Thus Y is metrizable (Theorem 4.3.14), completely metrizable, and homeomorphic to a complete separable metric space (Corollary 4.3.25). □

Corollary 4.4.13. (Π^1_1-CA_0) If a Hausdorff countably based UF space is strongly locally compact then it is metrizable, completely metrizable, and homeomorphic to a complete separable metric space.

Proof. Let P be a countable poset such that $X = \text{UF}(P)$ is strongly locally compact. We reason in Π^1_1-CA_0. By Theorem 4.4.10, X embeds in its one-point compactification Y. By Lemma 4.4.11, Y is regular. Thus Y is metrizable (Theorem 4.3.14), completely metrizable and homeomorphic to a complete separable metric space (Theorem 4.3.27). □

The Reverse Mathematics strength of Corollaries 4.4.12 and 4.4.13 is not known. All of the reversals we have presented in this thesis have used noncompact MF spaces. This motivates the following question.

Open Problem 4.4.14. Determine the Reverse Mathematics strength of the proposition that every locally compact Hausdorff countably based MF space is metrizable. The same question may be asked for UF spaces. We ask similar questions with "metrizable" replaced with "weakly metrizable," "completely metrizable," and "homeomorphic to a complete separable metric space." We ask similar questions if X is compact.

A open subset U of a metric space is *totally bounded* if for each $n \in \mathbb{N}$ there is a finite set D_n of points in U such that every point in U is within distance $1/n$ of some point in D_n. A metric space is *locally totally bounded* if every point is contained in a totally bounded open ball. The next theorem answers a question of of Hirst [Hir93], who asks if it is provable in second-order arithmetic that every locally totally bounded complete separable metric space has a one-point compactification.

Theorem 4.4.15. (Π^1_1-CA_0) Let $\widehat{A}$ be a complete separable metric space which is locally totally bounded. Then there is an embedding of $\widehat{A}$ into its one-point compactification.

Proof. The set of $\langle a, r\rangle \in A \times \mathbb{Q}^+$ such that $B(a, r)$ is totally bounded is arithmetically definable; thus we may form a sequence $\langle\langle a_i, r_i\rangle \mid i \in \mathbb{N}\rangle$ such that $B(a_i, r_i)$ is totally bounded for each $i \in \mathbb{N}$. WKL_0 proves that a complete separable metric space is compact if and only if it is complete and totally bounded (see [Sim99, Theorem IV.1.5]). Thus ACA_0 proves that a locally totally bounded complete separable metric space is strongly locally compact. Within ACA_0, we may form a poset P such that $\widehat{A} \cong \mathrm{UF}(P)$; $\mathrm{UF}(P)$ will also be strongly locally compact. We apply Corollary 4.4.10 to find a countable poset Q such that $\mathrm{UF}(Q)$ is the one-point compactification of $\widehat{A}$. By Lemma 4.4.11, $\mathrm{UF}(Q)$ is regular. The rest of the corollary follows from Corollary 4.3.27. □

Hirst [Hir93] has proved the following special case of Theorem 4.4.15 in ACA_0. We prove it in ACA_0^+ as a corollary of our results on one-point compactifications of UF spaces.

Corollary 4.4.16. (ACA_0^+) Every countable closed locally totally bounded subset of a complete separable metric space has a one-point compactification.

Proof. Let $\widehat{A} = \langle z_i\rangle$ be a countable complete separable metric space. Working in ACA_0, we can form the poset representation $P_{\widehat{A}}$. Moreover, ACA_0 proves that $UF(P_{\widehat{A}})$ is strongly regular; the key point is that we may quantify over the points of $P_{\widehat{A}}$ with number quantifiers because $\widehat{A}$ is countable. The rest of the corollary follows from Lemma 4.3.32 using ACA_0^+. □

4.5 Cardinality and perfect sets

In this section, we formalize the results on cardinality of poset spaces which were proved in ZFC in Chapter 2. We also explore several perfect set theorems.

4.5.1 Definitions

We will prove cardinality dichotomy theorems for countably based Hausdorff UF and MF spaces. These theorems say that an appropriate kind of space has either as few points as possible (countably many) or as many points as possible (one for each subset of $\mathbb{N}$). Before we can prove these dichotomy theorems, we must first define the notions of cardinality involved. It is straightforward to define in RCA_0 what it means for a poset space to be countable.

Definition 4.5.1. (RCA$_0$) Let P be a countable poset and let X be UF(P) or MF(P). We say that X is *countable* if there is a sequence $\langle F_i \rangle \subseteq X$ such that for every filter $F \in X$ there is an $i \in \mathbb{N}$ such that $F = F_i$.

It is more difficult to say that a poset space has one point for each subset of $\mathbb{N}$, because we cannot directly construct an uncountable set of points. We know that there are as many paths through $2^{<\mathbb{N}}$ as there are sets of natural numbers. Recall that if $g \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$ then $g[n] = \langle g(0), \ldots, g(n) \rangle \in 2^{<\mathbb{N}}$.

Definition 4.5.2. (RCA$_0$) Let X be UF(P) or MF(P), where P is a countable poset. We say X has *continuum-many points* if there is a function $F \colon 2^{<\mathbb{N}} \to P$ such that

1. For any $g \in 2^{\mathbb{N}}$, if $n < m$ then $F(g[n]) \preceq F(g[m])$. In particular, the set $\{F(g[n]) \mid n \in \mathbb{N}\}$ is linearly ordered.
2. If $g, h \in 2^{\mathbb{N}}$ and $g[n] \neq h[n]$ then $F(g[n]) \perp F(h[n])$.

We pause to justify the previous definition. The map F which exists if a poset space X has continuum-many points resembles an embedding of the Cantor space $2^{\mathbb{N}}$ into X. The existence of such an embedding would prove, classically, that a space had at least as many points as $2^{\mathbb{N}}$ (and a space with a countable basis can have no more points than this). The function F might be viewed by someone "living in RCA$_0$" as evidence that such an embedding exists.

Remark 4.5.3. It is consistent with ZFC that there are subsets of Baire space which have cardinality $2^{\aleph_0}$ but which do not have continuum-many points in the sense of Definition 4.5.2.

Definition 4.5.4. (RCA$_0$) A *perfect set* in a poset space is a closed subset with no isolated points in the subspace topology. We say that a definable subset U of a poset space X *contains a perfect subset* if there is a perfect subset C of X such that $C \subseteq U$.

Definition 4.5.5. Let $\mathcal{X}$ be a class of topological spaces. The *perfect set theorem for* $\mathcal{X}$ is the proposition that every space in $\mathcal{X}$ is either countable or contains a perfect subset. Thus we may speak of the perfect set theorem for UF spaces, the perfect set theorem for closed subsets of MF spaces, etc. We may formalize certain perfect set theorems in second-order arithmetic.

The *perfect set theorem for countably based Hausdorff UF spaces* is the following proposition, expressed as a sentence in the language of second-order arithmetic. If P is a countable poset such that UF(P) is Hausdorff then either UF(P) is countable or there is a code for a continuous open bijection from the Cantor space into UF(P). We may define the *perfect set theorem for countably based Hausdorff MF spaces* and the *perfect set theorem for closed subsets of countably based Hausdorff MF spaces* similarly.

A classical theorem shows that, in ZFC, a subset of a complete metric space is contains a perfect set if and only if it contains a homeomorphic image of Cantor space. This classical theorem gives a simple sufficient condition for the existence of perfect sets in complete metric spaces. We obtain a similar characterization of perfect subsets of Hausdorff UF spaces in Theorem 4.5.22. In the remainder of Section 4.5.1, we will establish, in ACA_0, sufficient conditions for the existence of perfect subsets of countably based poset spaces.

Open Problem 4.5.6. Suppose that U is an uncountable perfect subset of a countably based Hausdorff MF space. Does ZFC prove that U contains a homeomorphic image of the Cantor space? If so, is it provable in Z_2?

Definition 4.5.7. (RCA_0) The *standard poset representation of the Cantor space* is the poset $2^{<\mathbb{N}}$, ordered by letting $\tau \preceq \sigma$ if $\sigma \subseteq \tau$.

Lemma 4.5.8. (RCA_0) Let X be a countably based poset space and let U be a definable subset of X. If h is a continuous injection from the (standard poset representation of) Cantor space to a closed subset C of U, then U contains a perfect subset.

Proof. Let h be a continuous injection from $2^{\mathbb{N}}$ to U, and let $C \subseteq U$ be the closed set which is the range of h. We only need to show that C has no isolated points. Let $x \in C$ and choose $p \in P$ with $x \in N_p$. By assumption, there is an $f \in 2^{\mathbb{N}}$ such that $h(f) = x$. Because h is continuous, there is an $n \in \mathbb{N}$ such that $h(g) \in N_p$ for any $g \in 2^{\mathbb{N}}$ such that $g[n] = f[n]$. We can effectively choose $f' \in 2^{\mathbb{N}}$ such that $f[n] = f'[n]$ and $f'(n+1) \neq f(n+1)$. Thus $h(f') \in N_p$, and $h(f') \neq h(f)$ because h is injective. This shows that $h(f)$ is not an isolated point of $h(2^{\mathbb{N}})$. □

Definition 4.5.9. (RCA_0) A function F from a countable poset P to a countable poset Q is *order preserving* if $p \preceq_P p' \Leftrightarrow F(p) \preceq_Q F(p')$ and $p \perp_P p' \Leftrightarrow F(p) \perp_Q F(p')$ for all $p, p' \in P$. If F is an order-preserving map from $2^{<\mathbb{N}}$ to P, we let

$$F(f) = \{F(\sigma) \mid \sigma \subseteq f\}$$

for each $f \in 2^{\mathbb{N}}$.

Lemma 4.5.10. (ACA_0) Let P be a countable poset and let X be $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$. Suppose that F is an order-preserving map from $2^{<\mathbb{N}}$ to P such that $F(f) \in X$ for all $f \in 2^{\mathbb{N}}$. Then there is a continuous injection h from $2^{\mathbb{N}}$ to X such that $h(f) = F(f)$ for all $f \in 2^{\mathbb{N}}$. Moreover, the range of h is a closed subset of U.

Proof. We let $H = \{\langle \sigma, p \rangle \in 2^{<\mathbb{N}} \times P \mid F(\sigma) = p\}$. Let h be the continuous function coded by H. It is clear that $h(f) = F(f)$ for all $f \in 2^{\mathbb{N}}$. In particular, h is a total continuous function. Because F is order preserving, h is an injection. It remains to show that the range of h is closed.

Claim: Each point $y \in X$ is in the range of h if and only if for every $p \in P$ with $y \in N_p$ there is a $\sigma \in 2^{<\mathbb{N}}$ such that $F(\sigma) \preceq p$. One direction of the proof is easy. If $F(f) = y$ then $y = \mathrm{ucl}(\{F(f[n]\} \mid n \in \mathbb{N}\}$.

To prove the other direction, suppose that for every $p \in P$ such that $y \in N_p$ there is a $\sigma \in 2^{<\mathbb{N}}$ such that $F(\sigma) \preceq p$. Choose $p, p' \in y$ and choose $\sigma, \sigma' \in 2^{<\mathbb{N}}$ such that $F(\sigma) \preceq p$ and $F(\sigma') \preceq p'$. Because p and p' are compatible and F is order preserving, σ and σ' are compatible. This shows that the set $f = \{\sigma \in 2^{<\mathbb{N}} \mid F(\sigma) \in y\}$ is linearly ordered. Suppose that f is finite. Then there is a σ such that $y = \mathrm{ucl}(\{F(\sigma)\}$. Thus $N_{F(\sigma)} = \{y\}$. This means that any two extensions of $F(\sigma)$ in P are compatible. This is impossible, because F is order preserving and thus $F(\sigma \frown \langle 0 \rangle) \perp F(\sigma \frown \langle 1 \rangle$. Thus f is infinite, that is, $f \in 2^{\mathbb{N}}$. Clearly, $F(f) = y$. This finishes the proof of the claim.

We now form the coded open set

$$V = \{q \in P \mid \neg \exists \sigma \in 2^{<\mathbb{N}} \, [F(\sigma) \preceq q]\}$$

by arithmetical comprehension. The claim above shows that a point $y \in X$ is in N_V if and only if y is not in the range of h. Thus the range of h is closed. □

Remark 4.5.11. Lemma 4.5.10 has a more straightforward proof in ZFC, which uses the fact that $2^{\mathbb{N}}$ is compact.

We now give the first sufficient condition for the existence of perfect subsets of countably based poset spaces.

Theorem 4.5.12. (ACA_0) Let P be a countable poset and let X be $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$. Let U be a definable subset of X. Suppose that there is an order-preserving map F from $2^{<\mathbb{N}}$ to P such that $F(f) \in U$ for all $f \in 2^{\mathbb{N}}$. Then U has a perfect subset.

Proof. Combine Lemmas 4.5.8 and 4.5.10. □

We will develop one more sufficient condition for the existence of perfect sets.

Definition 4.5.13. Let X be a countably based poset space. We may identify X with a subset of $2^{\mathbb{N}}$ by taking characteristic functions. Let U be a definable subset of X. Let T be a perfect subtree of $2^{<\mathbb{N}}$. We that T codes a *perfect subset of X in the Baire topology* if for each $f \in [T]$ the set x_f whose characteristic function is f is a filter in X, and for distinct $f, g \in [T]$ we have $x_f \neq x_g$.

Lemma 4.5.14. (ACA_0) Let X be a Hausdorff countably based poset space, and let U be a definable subset of X. Suppose that there is a perfect subset of U in the Baire topology. Then U has a perfect subset.

Proof. Let $\langle p_i \rangle$ be the standard enumeration of P. Let T code a perfect subset of U in the Baire topology. Only countably many $f \in [T]$ may have the property that there is an $k \in \mathbb{N}$ such $f(k) = 0$ that for all $n > k$; each such f corresponds to a minimal element of the poset. We may effectively find a perfect subtree of T such that no path through the subtree this property. We replace T by such a perfect subtree if necessary.

We construct an order-preserving map $h\colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ by induction. Let h send the empty sequence to itself. Suppose h is defined on $\sigma \in 2^{<\mathbb{N}}$. We may canonically choose distinct paths $f_0, f_1 \in [T]$ such that $\sigma \subseteq f_m$ for $m = 0, 1$. Because f_0 and f_1 are distinct paths through T, they correspond to distinct filters in X. Because X is Hausdorff, there are $k_0, k_1 > \operatorname{lh}(\sigma)$ such that $f_0(k_0) = 1$, $f_1(k_1) = 1$, and the poset elements p, p' represented by k_0 and k_1 respectively are incompatible. Let $h(\sigma \frown \langle m \rangle) = k_m$ for $m = 0, 1$. This completes the construction of h. It is straightforward to verify that the map $F\colon \sigma \mapsto p_{\operatorname{lh}(h(\sigma))}$ is an order-preserving map from $2^{<\mathbb{N}}$ to P and $F(f) \in U$ for all $f \in 2^{\mathbb{N}}$. Thus, by Theorem 4.5.12, U has a perfect subset. □

4.5.2 Perfect sets and UF spaces

In this section, we formalize the poset star game of Section 2.2.2 to prove the perfect set theorem for countably based Hausdorff UF spaces. We show that the perfect set theorem for countably based Hausdorff UF spaces is equivalent to ATR_0 over ACA_0. We also show that the perfect set theorem for analytic subsets of Hausdorff UF spaces is equivalent to ATR_0 over ACA_0.

We begin by stating some definitions from [Sim99, Section V.8]. A *strategy for* I is a function s_{I} from $\bigcup_{n\in\mathbb{N}} \mathbb{N}^{2n}$ to $\mathbb{N}$. A *strategy for* II is a function s_{II} from $\bigcup_{n\in\mathbb{N}} \mathbb{N}^{2n+1}$ to $\mathbb{N}$. If s_{I} is a strategy for I and s_{II} is a strategy for II, we define $s_{\mathrm{I}} \otimes s_{\mathrm{II}} \in \mathbb{N}^{\mathbb{N}}$ to be the unique function such that

$$(s_{\mathrm{I}} \otimes s_{\mathrm{II}})(2n+1) = s_{\mathrm{I}}\big((s_{\mathrm{I}} \otimes s_{\mathrm{II}})[2n]\big),$$
$$(s_{\mathrm{I}} \otimes s_{\mathrm{II}})(2n+2) = s_{\mathrm{II}}\big((s_{\mathrm{I}} \otimes s_{\mathrm{II}})[2n+1]\big),$$

for all $n \in \mathbb{N}$. Thus $s_{\mathrm{I}} \otimes s_{\mathrm{II}}$ is the play which results when I plays strategy s_{I} and II plays strategy s_{II}.

Let $\Phi(X)$ be an L_2-formula with one free set variable. We say that Φ is *determined* if

$$\exists s_{\mathrm{I}} \forall s_{\mathrm{II}} \Phi(s_I \otimes s_{\mathrm{II}}) \lor \exists s_{\mathrm{II}} \forall s_{\mathrm{I}} \neg\Phi(s_{\mathrm{I}} \otimes s_{\mathrm{II}}), \tag{4.5.1}$$

where s_{I} ranges over strategies for I and s_{II} ranges over strategies for II. Because $s_{\mathrm{I}} \otimes s_{\mathrm{II}}$ is definable from s_{I} and s_{II} by primitive recursion, we see that that (4.5.1) is expressible in the language of second-order arithmetic. The following theorem is due to Steel [Ste76]; a proof may also be found in [Sim99, Section V.9].

Theorem 4.5.15 (Steel). ATR_0 proves that every Σ^0_1 or Π^0_1 formula with one free set variable is determined.

The theorem may be rephrased to say that ATR_0 proves that closed and open subsets of $\mathbb{N}^{\mathbb{N}}$ are determined.

We next define a formula $\Phi_{\mathrm{c}}(f, P)$ which says that f is a winning play for I in the poset star game on P:

$$\begin{aligned}\Phi_{\mathrm{c}}(f, P) \equiv\ & \forall n(p^n_0 \in P \land p^n_1 \in P) \land \forall n(p^n_0 \perp_P p^n_1) \\ & \land \forall n(f(2n+2) \in \{0,1\} \land p^{n+1}_0 \preceq p^n_{f(2n+2)} \land p^{n+1}_1 \preceq p^n_{f(2n+2)}),\end{aligned}$$

where p^n_i is an abbreviation for the element at position $(f(2n+1))_i$ in the standard enumeration of P. It is straightforward to see that Φ_{c} can be written as a Π^0_1 formula. Thus ATR_0 proves that Φ_{c} is determined.

Lemma 4.5.16. (RCA_0) Let P be a countable poset such that $\mathrm{UF}(P)$ is Hausdorff. Suppose that $\exists s_{\mathrm{I}} \forall s_{\mathrm{II}} \Phi_{\mathrm{c}}(s_{\mathrm{I}} \otimes s_{\mathrm{II}}, P)$. Then $\mathrm{UF}(P)$ has continuum-many points.

Proof. We must define a map F from $2^{\mathbb{N}}$ to P with the appropriate properties. It is clear that any $\tau \in 2^{<\mathbb{N}}$ determines the initial segment of a play s_{II} for II, and this play is determined by τ for $|\tau|$ stages. We may thus define

$F(\tau)$ to be the the element of P chosen by II at stage $|\tau|$ of the game; this is uniquely determined by τ. The desired properties of F follow directly from the definition of Φ_{c}. $\square$

Lemma 4.5.17. (ATR_0). Let P be a countable poset such that $\mathrm{UF}(P)$ is Hausdorff. Suppose that $\exists s_{\mathrm{II}} \forall s_{\mathrm{I}} \neg \Phi_{\mathrm{c}}(s_{\mathrm{I}} \otimes s_{\mathrm{II}}, P)$. Then $\mathrm{UF}(P)$ is countable.

We prove this lemma by formalizing the proof of Lemma 2.3.26 in second-order arithmetic. We first introduce some terminology which will be required in the proof.

Let s_{II} be a strategy for II. We define a *position* to be a sequence $\pi = \langle\langle p_0^1, p_1^1\rangle, \dots \langle p_0^k, p_1^k\rangle\rangle$ such that $p_0^i \perp p_1^i$ for $i \leq k$ and π respects the strategy s_{II} in the sense that $p_j^{i+1} \leq p^i_{s_{\mathrm{II}}(\pi[i])}$ for $j \in \{0,1\}$ and $i < j$. A position π is *consistent* with $x \in \mathrm{UF}(P)$ if $x \in N_{s_{\mathrm{II}}(\pi)}$.

Lemma 4.5.18. (ACA_0) Let P be a countable poset such that $\mathrm{UF}(P)$ is Hausdorff. Suppose that $\exists s_{\mathrm{II}} \forall s_{\mathrm{I}} \neg \Phi_{\mathrm{c}}(s_{\mathrm{I}} \otimes s_{\mathrm{II}}, P)$. Then for every $x \in \mathrm{UF}(P)$ there is a position π consistent with x which cannot be extended to a longer position consistent with x.

Proof. The proof is by contradiction. If every position consistent with x extends to a longer position consistent with x, then we can compute a strategy s_{I} for I as follows. To begin the game, I finds a position consistent with x. At stage n, player I finds an extension π of the current position such that π is consistent with x. Note that there is a Π^0_1 formula $\phi(\pi, x)$ which holds if and only if π is a position consistent with x. Thus ACA_0 suffices to define the strategy s_{I} just described.

We have defined s_{I} such that $\Phi_{\mathrm{c}}(s_{\mathrm{I}} \otimes s_{\mathrm{II}}, P)$ holds. This is impossible, because s_{II} is a winning strategy for II. $\square$

We say that a position π is *maximal for* $x \in \mathrm{UF}(P)$ if π is consistent with x and π cannot be extended to a longer position consistent with x. Thus Lemma 4.5.18 may be restated as: every x has a maximal position π.

If x, y are distinct points in $\mathrm{UF}(P)$ then no position π can be maximal for both x and y. For there are neighborhoods N_p and N_q of x and y, respectively, such that p and q both extend π. Thus $\pi \frown \langle p, q\rangle$ is an extension of π which is consistent with x or y.

Proof of Lemma 4.5.17. We have already shown: for every point in $\mathrm{UF}(P)$ there is a maximal position consistent with x, and no position is maximal for two points of $\mathrm{UF}(P)$. The set of (finite) positions is a countable set. There is thus an injective map $\mathrm{UF}(P) \to \mathbb{N}$. We know that, classically, this implies that UF(P) is a countable set. To show that UF(P) is countable in second-order arithmetic, however, we must explicitly demonstrate a function from $\mathbb{N}$ onto UF(P). In the present case, this is complicated by the fact that the set

$$M = \{\pi \mid \exists x \in \mathrm{UF}(P)[\pi \text{ is maximal for } x]\}$$

is not known to be arithmetical. We will require several nontrivial consequences of ATR_0 to complete the proof.

The relation

$$\Psi(\pi, x) \equiv \pi \text{ is maximal for } x \text{ and } x \in \mathrm{UF}(P)$$

is easily seen to be arithmetical; we have shown that for every π there is at most one x such that $\Psi(\pi, x)$. Thus ATR_0 proves that the set M exists; see [Sim99, Theorem V.5.2].

For each $\pi \in M$ there is a unique $x_\pi \in \mathrm{UF}(P)$ such that π is a maximal position for x_π. We must construct the map $\pi \mapsto x_\pi$. To do so, we will use the Σ^1_1 axiom of choice. This axiom scheme includes each sentence of the form

$$\forall n \exists X \phi(n, X) \Rightarrow \exists Y \forall n \phi(n, (Y)_n)$$

in which ϕ is a Σ^1_1 formula with the free variables shown. ATR_0 proves every instance of this scheme. We will use the instance with the formula $\phi(n, x)$ which says that $x \in \mathrm{UF}(P)$ and π_n is a maximal position for x, where $\langle \pi_n \mid n \in \mathbb{N}\rangle$ is the canonical enumeration of M. We thus obtain a set Y such that $(Y)_n \in \mathrm{UF}(P)$ for all n and, moreover, for all $x \in \mathrm{UF}(P)$ there is an n such that $x = (Y)_n$. The desired n can be found by choosing π, a maximal condition for x, and then finding n such that $\pi_n = \pi$. □

The next lemma says, roughly, that if C is a countable subset of $2^\mathbb{N}$ then we may effectively find a perfect subset of $2^\mathbb{N} \setminus C$

Lemma 4.5.19. (RCA_0) Suppose that $C = \langle f_i \mid i \in \mathbb{N}\rangle$ is a sequence of descending sequences in $2^{<\mathbb{N}}$ such that for all i either f_i is always strictly descending or there is some n such that $f_i[n]$ is strictly descending and $f_i(k) = f_i(n)$ for all $k > n$. Then there is an order-preserving injection F from $2^{<\mathbb{N}}$ to $2^{<\mathbb{N}}$ such that for any $g \in 2^\mathbb{N}$ we have $F[g] \notin C$.

Proof sketch. We construct F by induction. At stage n, we define F on all sequences of length n. We ensure at stage n that if $f_n[n]$ is strictly descending then none of the nodes in $F \upharpoonright 2^n$ is consistent with $f_n[n]$ (we may make $F(\tau)$ longer than τ). This construction ensures that there is no path g such that $F[g] = f_n$. □

Lemma 4.5.20. (ACA_0) If P is a countable poset such that $\mathrm{UF}(P)$ has continuum-many points then there is an order-preserving map F from $2^{<\mathbb{N}}$ to P such that $F(f) \in \mathrm{UF}(P)$ for each $f \in 2^{\mathbb{N}}$ and $F(f) \neq F(g)$ for distinct $f, g \in 2^{\mathbb{N}}$.

Proof. Let $F\colon 2^{<\mathbb{N}} \to P$ be a map which shows that $\mathrm{UF}(P)$ has continuum-many points. Fix $p \in P$ and $\tau \in 2^{<\mathbb{N}}$. Because $F(\tau \frown (0)) \perp F(\tau \frown (1))$, it cannot be that $p \preceq F(\tau \frown (0))$ and $p \preceq F(\tau \frown (1))$. Thus the set $\{\tau \mid p \preceq F(\tau)\}$ is linearly ordered (the set may be finite or infinite). Define $f_p\colon \mathbb{N} \to 2^{<\mathbb{N}}$ such that

$$f_p(0) = \langle\,\rangle$$

$$f_p(n+1) = \begin{cases} f_p(n) \frown (0) & \text{if } p \preceq f_p(n) \frown (0) \\ f_p(n) \frown \langle 1 \rangle & \text{if } p \preceq f_p(n) \frown (1) \\ f_p(n) & \text{otherwise} \end{cases}$$

RCA_0 proves that the sequence $C = \langle f_p \rangle$ exists, because the definition of f_p is uniform. Note that for every $g \in 2^{\mathbb{N}}$ if $p \preceq F(g[n])$ for all n then $f_p = f$. Thus we have proved: RCA_0 shows that the collection of paths through $2^{<\mathbb{N}}$ which map to bounded filters under F is countable, and the enumeration C includes all such paths.

We now apply Lemma 4.5.19. Let $H\colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ be such that $H[g] \notin C$ for all $g \in 2^{\mathbb{N}}$. Now $F \circ H\colon 2^{<\mathbb{N}} \to P$ is an order-preserving map, and for all $g \in 2^{\mathbb{N}}$ we see that $(F \circ H)[g] \in \mathrm{UF}(P)$. □

Lemma 4.5.21. (ACA_0) For each perfect subset C of the standard poset representation of $2^{\mathbb{N}}$ there is a perfect subtree T of $2^{<\mathbb{N}}$ such that $[T] \subseteq C$.

Proof. Let $U = \langle \tau_i \rangle$ be the open set complementary to C. Let T_0 be the set of $\sigma \in 2^{<\mathbb{N}}$ such that $\sigma \not\preceq \tau_i$ for all $i \in \mathbb{N}$. It is clear that $[T_0] = C$. Working in ACA_0, we may form a subtree T of T_0 such that T has no dead ends and $[T] = [T_0]$. The tree T satisfies the conclusion of the lemma. □

Theorem 4.5.22. The following are equivalent over ACA_0:

1. ATR_0
2. Let P be a countable poset such that $\mathrm{UF}(P)$ is a Hausdorff space. Then either $\mathrm{UF}(P)$ is countable or $\mathrm{UF}(P)$ contains a perfect subset.

Proof. We first prove (2) in ATR_0. We know that ATR_0 proves that Φ_c is determined. If I has a winning strategy, then $\mathrm{UF}(P)$ has continuum-many points (Lemma 4.5.16. Thus, by Lemmas 4.5.20 and 4.5.10, $\mathrm{UF}(P)$ has a perfect subset. If II has a winning strategy, then $\mathrm{UF}(P)$ is countable, as shown in Lemma 4.5.17.

For the reverse implication (2)$\Rightarrow$(1), we use the fact that ATR_0 is equivalent over RCA_0 to the statement "every subtree of $2^{<\mathbb{N}}$ either has countably many paths or has a perfect subtree." Let T be a subtree of $2^{<\mathbb{N}}$. Give T the poset order as a subposet of $2^{<\mathbb{N}}$. Thus $\mathrm{UF}(P)$ has one filter for each path through T and one filter for each dead end of T. If $\mathrm{UF}(P)$ is countable then $[T]$ is countable as well. Suppose that $\mathrm{UF}(P)$ has a perfect subset. Then, by Lemma 4.5.21, there is a perfect subtree of T. □

We next show that the perfect set theorem for analytic subsets of countably based Hausdorff UF spaces is equivalent to ATR_0 over ACA_0.

Definition 4.5.23. (RCA_0) Let P be a countable poset. A subset W of $\mathrm{UF}(P)$ is called *analytic* if there is a Σ^1_1 formula $\Psi(x)$ with one free set variable such that $\forall x[\Psi(x) \Leftrightarrow x \in W]$. We allow Ψ to have set parameters. We identify analytic subsets of UF spaces with the extensions of the formulas that define them.

We omit the straightforward proof of the following proposition (compare Definition V.1.5 and Theorem V.1.7 in [Sim99]).

Proposition 4.5.24. (RCA_0) Suppose that X and Y are countably based poset spaces and there is a continuous function $f\colon X \to Y$. The range of f is an analytic subset of Y.

The perfect set theorem for analytic subsets of countably based Hausdorff UF spaces is the following proposition, expressed as an L_2 sentence. Every analytic subset of a countably based Hausdorff UF space is either countable or has a perfect subset.

Theorem 4.5.25. The following are equivalent over ACA_0:

1. ATR_0.
2. The perfect set theorem for analytic subsets of countably based Hausdorff UF spaces.
3. The perfect set theorem for countably based Hausdorff UF spaces.

Proof. We first show that ATR_0 proves the perfect set theorem for analytic subsets of countably based Hausdorff UF spaces. ATR_0 proves the perfect set theorem for analytic subsets of $2^{\mathbb{N}}$; see Theorem V.4.3 and Theorem V.1.7 of [Sim99]. Thus an analytic subset W of a countably based Hausdorff UF space is either countable or has a perfect subset in the Baire topology. In the latter case, W has a perfect subset in the poset topology by Lemma 4.5.14.

Because $\mathrm{UF}(P)$ is an arithmetically definable subset of $2^{\mathbb{N}}$ (identifying filters with characteristic functions), the perfect set theorem for analytic subsets of countably based Hausdorff UF spaces implies the perfect set theorem for countably based Hausdorff UF spaces.

We showed in Theorem 4.5.22 that the perfect set theorem for countably based Hausdorff UF spaces implies ATR_0 over ACA_0. We apply that theorem now to complete the proof. □

4.5.3 Perfect sets and MF spaces

In this section, we first consider a cardinality dichotomy theorem for countably based Hausdorff MF spaces. We then turn to the perfect set theorem for closed subsets of countably based Hausdorff MF spaces. We show that this theorem is independent of ZFC and give a characterization of its strength.

Theorem 4.5.26. The cardinality dichotomy for MF spaces is the proposition that every countably based Hausdorff MF space is either countable or has continuum-many points. This proposition is provable in Π^1_2-CA_0 and implies ATR_0 over ACA_0.

Proof. The reversal in Theorem 4.5.22 shows that the dichotomy theorem for MF spaces implies ATR_0 over ACA_0. We sketch the proof that the dichotomy theorem for MF spaces is provable in Π^1_2-CA_0. Π^1_2-CA_0 proves that Φ_c is determined, and the proof of Lemma 4.5.16 shows that if I has a winning strategy then $\mathrm{MF}(P)$ has continuum-many points. It only remains to show that if II has a winning strategy then $\mathrm{MF}(P)$ is countable.

Suppose II does have a winning strategy. The proof of Lemma 4.5.18 shows that for every $x \in \mathrm{MF}(P)$ there is a maximal position. Because $\mathrm{MF}(P)$ is Hausdorff, no position is maximal for two distinct points. We may use Π^1_2 comprehension to form the set M of all positions π for which there is an $x \in \mathrm{MF}(P)$ such that π is a maximal position for x. We may then use Π^1_1 choice (provable in Π^1_2-CA_0; see [Sim99, Theorem VII.6.9]) to choose a sequence $\langle x_\pi \in \mathrm{MF}(P) \mid \pi \in M\rangle$ such that each position $\pi \in M$ is a maximal position for x_π. Thus $\mathrm{MF}(P) = \langle x_\pi \mid \pi \in M\rangle$ is countable. □

Definition 4.5.27. The *perfect set theorem for coanalytic sets* is the following scheme of L_2 sentences. For each Π^1_1 formula $\Psi(x)$ with one free set variable (but possibly with set parameters), either the set $\{x \mid \Psi(x)\} \subseteq \mathbb{N}^{\mathbb{N}}$ is countable or it has a perfect subset.

Lemma 4.5.28. (ACA_0) The perfect set theorem for coanalytic sets implies the perfect set theorem for closed subsets of countably based Hausdorff MF spaces.

Proof. Let P be a countable poset such that $\mathrm{MF}(P)$ is Hausdorff, and let C be a closed subset of $\mathrm{MF}(P)$. Define $\Phi(x) \equiv \mathrm{Filt}(x) \wedge x \in \mathrm{MF}(P) \wedge x \in C$. It is clear that Φ is a Π^1_1 formula. An instance of the perfect set theorem for Π^1_1 sets shows that either the set defined by Φ is countable or is has a perfect subset. If the set defined by Φ is countable then C is, trivially, countable as well. Otherwise, there is a perfect subtree T of $2^{<\mathbb{N}}$ such that for $\Phi(x)$ holds for all $x \in [T]$. Then, by Lemma 4.5.14, there is a perfect subset of $\mathrm{MF}(P)$. □

Lemma 4.5.29. (ACA_0) The perfect set theorem for closed subsets of countably based Hausdorff MF spaces implies the perfect set theorem for coanalytic sets.

Proof. Let S be a Π^1_1 subset of $\mathbb{N}^{\mathbb{N}}$. Apply Theorem 4.3.35 to obtain a poset P such that $\mathrm{MF}(P)$ is Hausdorff and there is a closed set $C \subseteq \mathrm{MF}(P)$ such that $S \cong C$. The perfect set theorem for closed subsets of countably based Hausdorff MF spaces implies that C is either countable or there is a code for a perfect subset of C. Let U be the open subset of $\mathrm{MF}(P)$ complementary to C. It is straightforward to convert U into an open subset U' of $\mathbb{N}^{\mathbb{N}}$. The complement of U' in $\mathbb{N}^{\mathbb{N}}$ is thus a perfect subset of S. □

The proof of the next theorem follows immediately from the previous two lemmas.

Theorem 4.5.30. The perfect set theorem for coanalytic sets is equivalent over ACA_0 to the perfect set theorem for closed subsets of countably based Hausdorff MF spaces.

Corollary 4.5.31. The perfect set theorem for closed subsets of countably based Hausdorff MF spaces is independent of ZFC and thus independent of the axioms of full second-order arithmetic.

In order to draw another corollary from Theorem 4.5.30, we survey some basic facts about Gödel's constructible hierarchy and its formalization in second-order arithmetic. Gödel proved that for any set X there is a unique smallest model $L(X)$ of ZFC containing X and all of the ordinals. This model is stratified: $L(X) = \bigcup_{\alpha \in \mathrm{ON}} L_\alpha(X)$, where ON denotes the class of ordinal numbers. Gödel showed, moreover, that every subset of $\mathbb{N}$ which is in $L(X)$ is in $L_\alpha(X)$ for some $\alpha < \aleph_1^{L(X)}$, where $\aleph_1^{L(X)}$ denotes the smallest ordinal which is not countable in $L(X)$.

Simpson [Sim99, Section VII.4] presents a thorough formalization of the theory of $L(A)$, for $A \subseteq \mathbb{N}$, in second-order arithmetic. In particular, the predicates $Y \in L(A)$ and $Y \in L_\alpha(A)$ are definable in ATR_0. There is thus an L_2 sentence which says that $\aleph_1^{L(A)}$ is countable, for $A \in P(\mathbb{N})$. The sentence says that there is a countable ordinal α such that for all $B \subseteq \mathbb{N}$, if $B \in L(A)$ then $B \in L_\alpha(A)$.

Theorem 4.5.32. (Π^1_1-CA_0) The perfect set theorem for coanalytic sets holds if and only if $\aleph_1^{L(A)}$ is countable for all $A \subseteq \mathbb{N}$.

Proof. A proof in ZFC is given by Mansfield and Weitkamp [MW85, Chapter 6]. The key tools in the proof are Kondo's Π^1_1 uniformization theorem and the Shoenfield Absoluteness Theorem. Simpson has also shown that Kondo's theorem [Sim99, Theorem VI.2.6] and the Shoenfield Absoluteness Theorem [Sim99, Theorem VII.4.14] are provable in Π^1_1-CA_0. Once these results have been established, it is possible to formalize the proof of Mansfield and Weitkamp in Π^1_1-CA_0. □

The following corollary is a straightforward combination of Theorems 4.5.30 and 4.5.32.

Corollary 4.5.33. (Π^1_1-CA_0) The perfect set theorem for closed subsets of countably based Hausdorff MF spaces holds if and only if $\aleph_1^{L(A)}$ is countable for all $A \subseteq \mathbb{N}$.

We have now determined the precise strength of the perfect set theorem for closed subsets of countably based Hausdorff MF spaces. It can be seen that the perfect set theorem for coanalytic sets implies the perfect set theorem for countably based Hausdorff MF spaces, but we do not know the strength of the latter perfect set theorem. Theorem 4.5.26 shows that the cardinality dichotomy theorem for countably based Hausdorff MF spaces is provable in Π^1_2-CA_0 and implies ATR_0.

Open Problem 4.5.34. Is the perfect set theorem for countably based Hausdorff MF spaces provable in ZFC? Is it provable in full second-order arithmetic?

References

[BHS87] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics*, Logic and combinatorics (Arcata, Calif., 1985), Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 125–156.

[Bro90] Douglas K. Brown, *Notions of closed subsets of a complete separable metric space in weak subsystems of second-order arithmetic*, Logic and computation (Pittsburgh, PA, 1987), Contemp. Math., vol. 106, Amer. Math. Soc., Providence, RI, 1990, pp. 39–50.

[Car37a] Henri Cartan, *Filtres et ultrafiltres*, CR Acad. Paris **205** (1937), 777–779 (French).

[Car37b] ———, *Théorie des filtres*, CR Acad. Paris **205** (1937), 595–598 (French).

[Coh76] Paul E. Cohen, *Products of Baire spaces*, Proc. Amer. Math Soc. **55** (1976), no. 1, 119–124.

[Cur03] Giovanni Curi, *Constructive metrisability in point-free topology*, Theoret. Comput. Sci. **305** (2003), no. 1-3, 85–109, Topology in computer science (Schloß Dagstuhl Seminar 231, 2000).

[FG82] M. P. Fourman and R. J. Grayson, *Formal spaces*, The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981), Stud. Logic Found. Math., vol. 110, North-Holland, Amsterdam, 1982, pp. 107–122.

[FK78] W. G. Fleissner and K. Kunen, *Barely Baire spaces*, Fund. Math. **101** (1978), no. 3, 229–240.

[GHK$^+$03] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous lattices and domains*, Encyclopedia of Mathematics and its Applications, vol. 93, Cambridge University Press, 2003.

[GKT60] R. O. Gandy, G. Kreisel, and W. W. Tait, *Set existence*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 577–582.

[GS53] David Gale and F. M. Stewart, *Infinite games with perfect information*, Contributions to the theory of games, vol. 2, Annals of Mathematics Studies, no. 28, Princeton University Press, Princeton, N. J., 1953, pp. 245–266.

[Hir93] Jeffry L. Hirst, *Embeddings of countable closed sets and reverse mathematics*, Arch. Math. Logic **32** (1993), no. 6, 443–449.

[Hjo02] Greg Hjorth, *A dichotomy theorem for turbulence*, J. Symbolic Logic **67** (2002), no. 4, 1520–1540.

[HKL90] L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928.

[HY88] John G. Hocking and Gail S. Young, *Topology*, second ed., Dover Publications Inc., New York, 1988.

[Joh82] Peter T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.

[Joh83] ______, *The point of pointless topology*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 1, 41–53.

[Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, 1995.

[Kel55] John L. Kelley, *General topology*, D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[KL97] Alexander S. Kechris and Alain Louveau, *The classification of hypersmooth Borel equivalence relations*, J. Amer. Math. Soc. **10** (1997), no. 1, 215–242.

[Law97] Jimmie Lawson, *Spaces of maximal points*, Math. Structures Comput. Sci. **7** (1997), no. 5, 543–555, Workshop on logic, domains, and programming languages (Darmstadt,1995).

[Mar03] Keye Martin, *Topological games in domain theory*, Topology Appl. **129** (2003), no. 2, 177–186.

[MK80] D. A. Martin and A. S. Kechris, *Infinite games and effective descriptive set theory*, Analytic sets, Academic Press, 1980, pp. 403–470.

[Mon76] J. Donald Monk, *Mathematical logic*, Springer-Verlag, New York, 1976, Graduate Texts in Mathematics, No. 37.

[MW85] Richard Mansfield and Galen Weitkamp, *Recursive Aspects of Descriptive Set Theory*, Oxford University Press, 1985, with a chapter by Stephen Simpson.

[Rog67] Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill Book Co., New York, 1967.

[Sac90] Gerald E. Sacks, *Higher recursion theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990.

[Sam03] Giovanni Sambin, *Some points in formal topology*, Theoret. Comput. Sci. **305** (2003), no. 1-3, 347–408, Topology in computer science (Schloß Dagstuhl Seminar 231, 2000).

[Sch98] Matthias Schröder, *Effective metrization of regular spaces*, Computability and Complexity in Analysis (K. Ko, A. Nerode, K. Pour-El, and K. Weihrauch, eds.), Informatik-Berichte, vol. 235, FernUniversität Hagen, 1998, pp. 63–80.

[Sim99] Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, 1999.

[Ste76] John R. Steel, *Determinateness and Subsystems of Analysis*, Ph.D. thesis, University of California at Berkeley, 1976.

[Sto37] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41** (1937), no. 3, 375–481.

Vita

Carl Mummert

Born: August 2, 1978 in Pittsburgh

Education

PhD	August 2005	The Pennsylvania State University Thesis Advisor: Stephen G. Simpson
BS	May 2000	Western Carolina University *summa cum laude*

Publications

1. *An Incompleteness Theorem for β_n-Models.*
   Carl Mummert and Stephen G. Simpson. Journal of Symbolic Logic 69 no. 2 (June 2004), 612–616.

2. *On the Reverse Mathematics of General Topology*
   PhD dissertation, August 2005.