Filters on computable posets

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Abstract

We explore the problem of constructing maximal and unbounded filters on computable posets. We obtain both computability results and reverse mathematics results.

A maximal filter is one that does not extend to a larger filter. We show that every computable poset has a Δ_2^0 maximal filter, and there is a computable poset with no Π_1^0 or Σ_1^0 maximal filter. There is a computable poset on which every maximal filter is Turing complete. We obtain the reverse mathematics result that the principle "every countable poset has a maximal filter" is equivalent to ACA_0 over RCA_0.

An unbounded filter is a filter which achieves each of its lower bounds in the poset. We show that every computable poset has a Σ_1^0 unbounded filter, and there is a computable poset with no Π_1^0 unbounded filter. We show that there is a computable poset on which every unbounded filter is Turing complete, and the principle "every countable poset has an unbounded filter" is equivalent to ACA_0 over RCA_0. We obtain additional reverse mathematics results related to extending arbitrary filters to unbounded filters and forming the upward closures of subsets of computable posets.

1 Introduction

In this paper, we study maximal and unbounded filters on computable posets. We obtain computability results and reverse mathematics results

regarding the existence of these filters.

We use the following terminology. A *poset* is a set P with a reflexive, antisymmetric, transitive relation $\preceq$. A poset $\langle P, \preceq \rangle$ is *computable* if P is a computable subset of $\mathbb{N}$ and $\preceq$ is a computable binary relation on P. A *filter* is a subset F of a poset such that F is upward closed (if $p \in F$ and $p \preceq q$ then $q \in F$) and for all $p, q \in F$ there is an $r \in F$ with $r \preceq p$ and $r \preceq q$. The entire poset is thus a filter if every pair of elements is compatible. A filter is *maximal* if it is not contained in a strictly larger filter. A filter F is *unbounded* if there is no $p \notin F$ with $p \preceq q$ for all $q \in F$.

In Section 2, we study maximal filters. We show that every computable poset has a Δ_2^0 maximal filter. This result is optimal: there is a computable poset with no Σ_1^0 or Π_1^0 maximal filter. There is also a computable poset P such that any maximal filter on P is Turing complete. We obtain a reverse mathematics result: the principle that every countable poset has a maximal filter is equivalent to ACA_0 over RCA_0.

In Section 3, we study unbounded filters. We show that every computable poset has a Σ_1^0 unbounded filter, and there is a computable poset with no Π_1^0 unbounded filter. There is a computable poset P such that every unbounded filter on P is Turing complete. We obtain two reverse mathematics results. The principle that every countable poset has an unbounded filter is equivalent to ACA_0 over RCA_0. We define enumerated filters, which are analogous to Σ_1^0 filters, and show that the principle "Every sequence of enumerated filters on countable posets extends to a sequence of unbounded enumerated filters" is equivalent to ACA_0 over RCA_0.

In Section 4, we show that the upward closure of a computable subset of a computable poset may be Turing complete. The principle that every subset of a countable poset has an upward closure is equivalent to ACA_0 over RCA_0.

There has been significant previous research on the computability aspects of linear and partial orders. Downey [1] gives a thorough description of many results. We note that Mummert has shown [2] that there is a computable poset with a computable filter F such that the complete Σ_1^1 set is one-one reducible to any extension of F to a maximal filter.

The following results appeared in the second author's PhD thesis [3]: Corollaries 2.2 and 4.2 and Theorems 3.1, 3.5, and 3.6. The remaining results are due to both authors.

1.1 Notation

We use the following standard notation from computability theory. W_e denotes the Σ_1^0 set with index e and $W_{e,s}$ denotes the subset of W_e which is enumerated in s or fewer steps. $0'$ denotes the canonical Σ_1^0 complete set. The symbol $\leq_{wtt}$ denotes weak (bounded) truth table reducibility of subsets of $\mathbb{N}$. Readers unfamiliar with these concepts may consult the texts by Rogers [4] or Soare [6].

Reverse mathematics is a program in mathematical logic which classifies theorems based on the set-existence (comprehension) axioms required to prove the theorems. This classification is made using subsystems of second-order arithmetic. In this paper, we use two subsystems of second-order arithmetic: RCA_0, which contains Δ_1^0 comprehension and Σ_1^0 induction, and ACA_0, which contains arithmetical comprehension and arithmetical induction. A complete definition of these subsystems, along with a complete description of the goals of reverse mathematics, is given by Simpson [5].

2 Maximal filters

A filter on a computable poset is maximal if it is not strictly contained in another filter on the poset.

Theorem 2.1. Every computable poset has a Δ_2^0 maximal filter.

Proof. Let P be a computable poset. Begin by forming the oracle $A = \{\langle p, q \rangle \in P \times P \mid \exists r (r \preceq p \wedge r \preceq q)\}$. This oracle is clearly Σ_1^0. Now we define a maximal filter $G = \{q_i \mid i \in \mathbb{N}\}$ on P inductively. At stage 0, let $q_0 = p_0$. At stage $i + 1$, consider element $p_{i+1} \in P$. If $\langle p_{i+1}, q_i \rangle \in A$ then we may effectively let q_{i+1} be a common extension of q_i and p_{i+1}. If $\langle p_{i+1}, q_i \rangle \notin A$, let $q_{i+1} = q_i$. It is not hard to show that F is a maximal filter on P. If element p_i is added to G then it is added no later than stage i; thus G is computable from A, so G is Δ_2^0. $\qquad\square$

Corollary 2.2. ACA_0 proves that every countable poset has a maximal filter.

Proof. The previous proof may be formalized in ACA_0, using Σ_1^0 comprehension to form A. $\qquad\square$

Theorem 2.3. There is a computable poset P such $0' \leq_{wtt} F$ for any maximal filter F on P.

Proof. Let K be any Σ_1^0 set. For each $k \in \mathbb{N}$, we construct a poset P_k inductively. P_k begins with two incompatible descending sequences. These sequences will eventually become compatible if $k \in K$; otherwise they will become two incompatible infinite descending sequences. If the sequences do not merge, then exactly one of the two maximal elements of the sequences will appear in any maximal filter, while if the sequences do merge then both maximal elements must appear in the unique maximal filter.

Formally, we let $P_k = \mathbb{N}$ and define the order $\preceq_k$ with the following rules:

$$
\begin{aligned}
2m &\preceq_k 2n & &\text{if } n \leq m, \\
2m + 1 &\preceq_k 2n + 1 & &\text{if } n \leq m, \\
2m &\preceq_k 2n + 1 & &\text{if } n \leq m \text{ and } k \in W_{e,m}, \\
2(m+1) + 1 &\preceq_k 2n & &\text{if } n \leq m \text{ and } 2m + 2 \preceq 2m + 1.
\end{aligned}
$$

Clearly $\preceq_k$ is a computable partial order. If $k \notin K$ then $2m \perp 2n + 1$ for all n, m, and no maximal filter can contain both 0 and 1. Otherwise, for all sufficiently large m, n we have $2m \preceq 2n + 1$ and $2m + 1 \preceq 2n$, every two elements are compatible, and the unique maximal filter contains 0 and 1.

The construction above is uniform in the sense that we may uniformly compute the order $\preceq_k$ from k. We may thus form the product poset P, which we now describe. The elements of P are finite sequences of natural numbers. We order P by putting $\bar{a} \preceq \bar{b}$ if and only if $|\bar{a}| \geq |\bar{b}|$ and $a_i \preceq_i b_i$ for all $i \leq |\bar{b}|$. Clearly the order on P is computable. Note that for any $k \in \mathbb{N}$ and any maximal filter F on P, the set $\pi_k F = \{n \in \mathbb{N} \mid \exists \bar{a} \in F(|\bar{a}| \geq k \wedge n = a_k)\}$ is a maximal filter on P_k.

Note that for each k, exactly one of the following options holds.

1. $0 \in \pi_k F$ or $1 \in \pi_k F$, but not both.

2. $0 \in \pi_k F$ and $1 \in \pi_k F$.

Beginning with $k = 0$ and proceeding inductively, we may effectively determine for each k which of the options holds using no more than 2 queries for each new k. For example, we first determine whether 0 or 1 is in $\pi_0 F$. Assume that $0 \in \pi_0 F$. Then we can determine whether $1 \in \pi_1 F$ by asking whether $\langle 0, 1 \rangle \in F$.

If the first option above holds for k then $k \notin K$. If the second option holds, then $k \in K$. Thus $K \leq_{wtt} G$ for any maximal filter G on P. The theorem follows by taking $K = 0'$. $\square$

Corollary 2.4. The principle that every countable poset has a maximal filter is equivalent to ACA_0 over RCA_0.

Proof. The proof of Theorem 2.3, relativized to an arbitrary oracle, may be formalized in RCA_0. $\qquad\square$

Theorem 2.5. There is a computable poset with no Σ^0_1 or Π^0_1 maximal filter.

Proof. We informally view our poset P as an infinite product of posets $\langle P_e \mid e \in \mathbb{N} \rangle$ and $\langle Q_e \mid e \in \mathbb{N} \rangle$. The construction of P_e ensures that W_e is not a maximal filter on P, and the construction of Q_e ensures that the complement of W_e is not a maximal filter on P.

For each e, the poset P_e is built as follows. We first construct an infinite descending sequence $\langle a_i \mid i \in \mathbb{N} \rangle$. Now we wait until an element a_i enters W_e. If no element a_i enters, then W_e is not a maximal filter on P_e. Whenever an element a_{i+1} enters W_e, we add an element b_i such that $b_i \prec a_i$ and $b_i \perp a_{i+1}$. Then we return to waiting. If no element of the form b_i is ever added to W_e, then the infinite descending sequence $\langle b_i \rangle$ generates a filter properly including W_e. If W_e ever includes any element of the form b_i, we stop adding any new elements of the form b_j to the poset. Now, because b_i was added, we know a_{i+1} is in W_e, and $b_i \perp a_j$ for all $j \geq i+1$ because we will add no more elements of the form b_j. Thus W_e is not a maximal filter on P_e.

For each e, the poset Q_e is built as follows. We construct two infinite descending sequences $\langle a_i \rangle$ and $\langle b_j \rangle$ as in the proof of Theorem 2.3. We wait to see whether W_e ever includes a_0 or b_0. If it never does, we keep the infinite descending chains incompatible, which means that the complement of W_e is not a filter on Q_e. If W_e ever includes a_0 or b_0, we cause the two chains to eventually be compatible, which means that the complement of W_e is not a maximal filter on Q_e.

To construct the poset P, we view each natural number as a code for a finite sequence of natural numbers, and thus view $P = \mathbb{N}$ as a product poset with one coordinate for each natural number. The order on coordinate $2i$ of P is similar to the order on poset P_i, and the order on coordinate $2i+1$ of P is similar to the order on Q_i. We watch the effects of each Σ^0_1 or Π^0_1 subset on the appropriate coordinate of P and use the descriptions of P_e and Q_e above to shape the order in that coordinate. Because the elements of P are finite sequences and the order relations on P_e and Q_e are uniformly computable, the order on P will be computable. $\qquad\square$

3 Unbounded filters

A filter F on a poset P is unbounded if there is no $p \in P \setminus F$ such that $p \preceq q$ for all $q \in F$.

Theorem 3.1. Every computable poset has a Σ_1^0 unbounded filter.

Proof. Let $P = \langle p_i \mid i \in \mathbb{N} \rangle$ be a computable poset. We describe an enumeration of an unbounded filter on P. At stage 0, let $q_0 = p_0$. At stage $n + 1$, if $p_{i+1} \leq q_i$ then let $q_{i+1} = p_{i+1}$; otherwise, let $q_{i+1} = q_i$. The upward closure of $\{q_i \mid i \in \mathbb{N}\}$ is easily seen to be a Σ_1^0 unbounded filter on P. $\qquad\square$

Theorem 3.2. There is a computable poset P such that $0' \leq_{wtt} F$ for any unbounded filter F on P.

Proof. We use the computable posets $\langle P_k, \preceq_k \rangle$ from the proof of Theorem 2.3. We form a computable poset P whose elements are sequences of the form $\langle n, a_0, a_1, \ldots, a_n \rangle$ where $n \in \mathbb{N}$ and $a_k \in P_k$ for each $k \leq n$. We order P by putting $\langle n, a_0, \ldots, a_n \rangle \prec \langle m, b_0, \ldots, b_m \rangle$ when $n > m$ and $a_i \prec_i b_i$ for each $i \leq m$. Clearly P is a computable poset; we call this the uniform product of the posets $\langle P_k \rangle$.

We now show that the projection of any unbounded filter on P to coordinate $k + 1$ is an unbounded filter on P_k. Clearly any unbounded filter on P is generated by an infinite strictly descending sequence $\langle r_i \rangle$ on P. Let p_i denote the element of P_k in coordinate $k + 1$ of r_i whenever this coordinate exists. The order relation on P ensures that $\langle p_i \rangle$ is an infinite strictly descending sequence on P_k. Any such sequence determines an unbounded filter on P_k.

Note that every unbounded filter on P_k is maximal; thus we have shown that any unbounded filter on P uniformly computes a maximal filter on P_k for each k. The remainder of the proof is similar to Theorem 2.3. $\qquad\square$

We obtain a corollary by formalizing the previous proof in RCA_0.

Corollary 3.3. The principle that every countable poset has an unbounded filter is equivalent to ACA_0 over RCA_0.

The next theorem is proved by applying the uniform product technique of Theorem 3.2 to the sequence of posets Q_e in the proof of Theorem 2.3.

Theorem 3.4. There is a computable poset with no Π_1^0 unbounded filter.

In RCA_0, we define an *enumerated filter* on a countable poset $\langle P, \preceq \rangle$ to be a function F from $\mathbb{N}$ to P whose range is a filter on P. That is, if p is in the range of F and $p \preceq q$ then q is in the range of F, and for all p, q in the range of F there is an r in the range of F such that $r \preceq p$ and $r \preceq q$. Note that a filter on P is definable by a Σ_1^0 formula if and only if RCA_0 is able to form the corresponding enumerated filter. We view an enumerated filter as a code for the set of poset elements it determines, writing $p \in F$ for an enumerated filter F if p is in the range of F. We say that an enumerated filter is unbounded if there is no $q \in P$ such that $q \prec p$ whenever $p \in F$. By formalizing the proof of Theorem 3.1, we may prove in RCA_0 that every countable poset has an unbounded enumerated filter.

Theorem 3.5. RCA_0 proves that every enumerated filter on a countable poset is contained in an unbounded enumerated filter on the poset.

Proof. Let F be an enumerated filter on a countable poset P. If F is unbounded then we are done. If F is bounded, let p be a lower bound for F and construct an unbounded enumerated filter containing p as in the proof of Theorem 3.1. $\square$

The proof of Theorem 3.5 is nonuniform in the sense that the noncomputable choice of whether a given filter is unbounded must be made. The next theorem suggests that no uniform proof of Theorem 3.5 is possible.

Theorem 3.6. The following are equivalent over RCA_0.

1. ACA_0.

2. If $\langle P_i \mid i \in \mathbb{N} \rangle$ is a sequence of countable posets and $\langle F_i \mid i \in \mathbb{N} \rangle$ is such that each F_i is an enumerated filter on P_i then there is a sequence $\langle G_i \mid i \in \mathbb{N} \rangle$ such that G_i is an enumerated unbounded filter on P_i extending F_i for each $i \in \mathbb{N}$.

Proof. It is not difficult to prove (2) in ACA_0. To prove that (2) implies ACA_0, we work in RCA_0. Let $f \colon \mathbb{N} \to \mathbb{N}$ be given; we will show that the range of f exists, which implies ACA_0. For each $e \in \mathbb{N}$, form a poset $\langle P_e, \preceq_e \rangle$ as follows. P_e has an infinite descending sequence $\langle a_i^e \mid i \in \mathbb{N} \rangle$ and one additional element b^e. We let $b^e \preceq_e a_i^e$ if there is no $j \leq i$ such that $f(j) = e$. For all i we have $a_i^e \not\preceq_e b^e$. It is clear that $\preceq_e$ is a partial order on P_e, and the sequence of posets $\langle \langle P_e, \preceq_e \rangle \mid e \in \mathbb{N} \rangle$ may be formed in RCA_0.

For each $e \in \mathbb{N}$ let $F_e = \{a_i^e \mid i \in \mathbb{N}\}$; the sequence $\langle F_e \mid e \in \mathbb{N} \rangle$ may be formed in RCA_0 as well. Apply (2) to form a sequence $\langle G_e \mid e \in \mathbb{N} \rangle$ such that

each G_e is an enumerated filter on P_e and $F_e \subseteq G_e$. Note that $b^e \in G^e$ if and only if there is no $i \in \mathbb{N}$ such that $f(i) = e$. There is thus a Σ_1^0 formula which tells if e is not in the range of f. Because the range of f has a trivial Σ_1^0 definition, we have shown the range of f is definable by a Δ_1^0 formula, which means that we may form this set in RCA_0. $\qquad\square$

4   Upward closures

The upward closure of a set $F \subseteq P$ is the set $\{q \in P \mid \exists p \in F(p \preceq q)\}$.

Theorem 4.1. There is a computable poset P with a computable linearly ordered subset A such that $0'$ is one-one reducible to the upward closure of F.

Proof. Let W_e be a Σ_1^0 complete set. We let P have an infinite descending sequence $\langle a_i \mid i \in \mathbb{N} \rangle$ and infinitely many pairwise incompatible elements $\{b_j \mid j \in \mathbb{N}\}$. We put $a_i \preceq b_j$ if $j \in W_{e,i}$, and $a_i \not\preceq b_j$ otherwise. We always have $b_j \not\preceq a_i$.

The poset just described is computable, as is the subset $A = \{a_i \mid i \in \mathbb{N}\}$. Clearly $j \in W_e$ if and only if b_j is in the upward closure of A. $\qquad\square$

Corollary 4.2. The following are equivalent over RCA_0.

1. ACA_0.

2. Every subset of a countable poset has an upward closure.

3. Every linearly ordered subset of a countable poset has an upward closure.

Proof. Because the upward closure of any set $A \subseteq P$ is Σ_1^0 definable from A, ACA_0 proves (2). Clearly (2) implies (3). The proof of Theorem 4.1 may be formalized in RCA_0 to show that (3) implies ACA_0. $\qquad\square$

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