Subsystems of second-order arithmetic between RCA_0 and WKL_0

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Abstract

We study the Lindenbaum algebra $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ of sentences in the language of second-order arithmetic which imply RCA_0 and are provable from WKL_0. We explore the relationship between Σ^1_1 sentences in $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ and Π^0_1 classes of subsets of ω. By applying a result of Binns and Simpson (*Arch. Math. Logic*, 2004) about Π^0_1 classes, we give a specific embedding of the free distributive lattice with countably many generators into $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$.

1 Introduction

In this paper, we consider the algebra $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ of finitely axiomatizable subsystems of second-order arithmetic between WKL_0 and RCA_0. These subsystems can be naturally identified with a sublattice of the Lindenbaum algebra of sentences of second-order arithmetic. We give a specific embedding of the free distributive lattice on countably many generators into these subsystems. The central lemma for this embedding result comes from a result of Binns and Simpson [2] on the lattice of Muchnik degrees of Π^0_1 classes.

One motivation for choosing the systems WKL_0 and RCA_0 is that WKL_0 is conservative over RCA_0 for Π^1_1 sentences. It will be seen that the statements appearing in the results are of a purely mathematical character. Simpson [7] has shown that, despite conservativity, it is possible to use Π^0_1 statements

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arising from Gödel's incompleteness theorems to construct subsystems between WKL_0 and RCA_0; see Remark 3.5. Moreover, incompleteness results imply that $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ is atomless, as described below.

We consider second-order arithmetic in the language $L_2 = \langle 0, 1, +, \times, =, <, \in \rangle$. The subsystem RCA_0 includes the first-order axioms of Peano arithmetic without induction, comprehension for Δ^0_1 formulas with parameters, and induction for Σ^0_1 formulas with parameters. The subsystem WKL_0 is RCA_0 plus weak König's lemma, which says that any subtree of $2^{<\mathbb{N}}$ is either finite or has a path. A thorough description of these systems is given by Simpson [6].

We let $\mathcal{A}$ denote the Lindenbaum algebra of (equivalence classes of) L_2 sentences without parameters. Two sentences ϕ and ψ are equivalent if $\vdash \phi \Leftrightarrow \psi$. The order on $\mathcal{A}$ is defined so that $\phi \leq \psi$ if and only if $\psi \vdash \phi$; this order clearly respects the equivalence relation. It is well known that $\mathcal{A}$ is a Boolean algebra with operations $\sup([\phi], [\psi]) = [\phi \wedge \psi]$ and $\inf([\phi], [\psi]) = [\phi \vee \psi]$. The monograph of Grätzer [3] gives additional information on lattice theory.

For any L_2 sentence ϕ we let ϕ^* be the sentence $\phi \wedge \Theta_{\mathsf{RCA}_0}$, where Θ_{RCA_0} is the canonical finite axiomatization of RCA_0. We define

$$\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0) = \{[\phi^*] \in \mathcal{A} \mid \mathsf{WKL}_0 \vdash \phi^*\}.$$

It is not difficult to see that an equivalence class $[\phi]$ is in $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ if and only if $\mathsf{WKL}_0 \vdash \phi$ and $\phi \vdash \mathsf{RCA}_0$. Moreover, $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ is a Boolean algebra. The complement of a sentence ϕ is the sentence $\Theta_{\mathsf{RCA}_0} \wedge (\neg\phi \vee \Theta_{\mathsf{WKL}_0})$, where Θ_{WKL_0} is the canonical finite axiomatization of WKL_0.

The algebra $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ is atomless, and is thus isomorphic to the canonical countable atomless Boolean algebra. This follows from the fact that an atom in the alegbra would also be an atom in the full Lindenbaum algebra $\mathcal{A}(\perp, \mathsf{RCA}_0)$ of finitely axiomatized subsystems of second order arithmetic above RCA_0. The algebra $\mathcal{A}(\perp, \mathsf{RCA}_0)$ has no atoms because it has no coatoms, which would be complete, consistent finitely axiomatized extensions of RCA_0. These cannot exist in light of Gödel's incompleteness theorem.

There are several well-known subsystems of second-order arithmetic in $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$. Simpson and Yu [8] introduced the system WWKL_0, which is closely related to the reverse mathematics of measure theory, and proved that it is strictly between RCA_0 and WKL_0. Recently, Ambos-Spies *et al.* [1] have shown that the subsystem DNR_0 is strictly weaker than WWKL_0; DNR_0 is known to be stronger than RCA_0.

In this paper, we study the overall structure of $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ rather than studying specific named subsystems. In Section 2, we explore the relationship between Π^0_1 classes and elements of $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$. In Section 3, we give a natural embedding of the free distributive lattice on ω generators.

2 Π^0_1 classes

In this section we explore the relationship between $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ and the nonempty Π^0_1 subsets of 2^ω. We identify each natural number e with a computable (possibly empty) tree $T_e \subseteq 2^{<\omega}$ and the corresponding Π^0_1 class $Q_e \subseteq 2^\omega$. It is clear that for each e there is a Σ^1_1 sentence which says "Q_e is nonempty." We let $S(e)$ be the subsystem of second-order arithmetic consisting of RCA_0 plus the sentence that Q_e is nonempty. For each $e \in \omega$, WKL_0 will prove the sentence $S(e)$ if and only if it proves that the tree T_e is infinite.

Remark 2.1. We will sometimes limit our consideration to those subsystems $S(e)$ that are provable from WKL_0. This is not a vacuous restriction, for Gödel's incompleteness theorem implies that there is an $e \in \omega$ such that both $S(e)$ and $\mathsf{WKL}_0 + \neg S(e)$ are consistent.

We will use the following notation relating to Medvedev and Muchnik reducibility. Let P, Q be any nonempty Π^0_1 classes. We write $P \leq_w Q$ if every element of Q computes an element of P, and we write $P \leq_M Q$ if there is a Turing functional F such that $F[Q] \subseteq P$. We write $P \equiv_M Q$ if $P \leq_M Q$ and $Q \leq_M P$, and define $\equiv_w$ similarly. Rogers [5], Binns and Simpson [2], and Simpson [7] give more information about Medvedev ($\leq_M$) and Muchnik ($\leq_w$) reducibility.

We formalize computability theory in (possibly nonstandard) models of RCA_0 by identifying Turing reducibility with relative Δ^0_1 definability. This identification is possible because there is a Σ^0_1 formula that RCA_0 proves to be universal. Using this definition of Turing reducibility, we translate the definitions of $\leq_M, \leq_w, \equiv_M, \equiv_w$ into RCA_0. Additional comments on this formalization of computability theory are given by Mytilinaios [4] and Simpson [7].

Our first theorem illustrates the relationship between the order relation on $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ and provable Muchnik reducibility. In the proof, and for the rest of this paper, if M is an L_2 structure we write $X \in M$ to mean that X is a set in the second-order part of M.

Theorem 2.2. For any $a, b \in \omega$, $S(b) \vdash S(a)$ if and only if $\mathsf{RCA}_0 \vdash Q_a \leq_w Q_b$. Thus $S(b)$ does not prove $S(a)$ if $Q_a \not\leq_w Q_b$.

Proof. Suppose $S(b) \vdash S(a)$. Let M be any model of $S(b)$ and choose $X \in M$ such that $M \models X \in Q_b$. Let $M' \subseteq M$ consist of those sets $Y \in M$ such that $M \models Y \leq_T X$. It is well known that M' will be a model of RCA_0; compare Theorem IX.1.8 of [6]. Moreover, $M' \models X \in Q_b$, so $M' \models S(b)$. By assumption, there is a $Z \in M'$ be such that $M' \models Z \in Q_a$; clearly $M \models Z \leq_T X$. This shows $M \models Q_a \leq_w Q_b$.

Now suppose $\mathsf{RCA}_0 \vdash Q_a \leq_w Q_b$. Let M be any model of $S(b)$ and let $M \models X \in Q_b$. Then by assumption there is a $Y \in M$ such that $M \models Y \in Q_a \wedge Y \leq_T X$. We conclude $M \models Q_a$, which shows $S(b) \vdash S(a)$. □

The previous theorem shows that Muchnik-incomparable Π^0_1 classes yield incomparable elements of $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$. The next theorem shows that even Medvedev-equivalent Π^0_1 classes may correspond to distinct elements of $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$.

Theorem 2.3. Let Q_a be a Π^0_1 class such that WKL_0 proves Q_a is nonempty. There is a Π^0_1 class Q_b such that $Q_a \equiv_M Q_b$ but $\mathsf{WKL}_0 \nvdash Q_b \leq_w Q_a$. Thus $S(a)$ does not prove $S(b)$.

Proof. Let $\phi(n, m)$ be a Σ^0_1 formula such that for each n there is a unique m with $\phi(n, m)$, but there is no primitive recursive function f such that $\forall n\, \phi(n, f(n))$. Let $g(n)$ be the function such that $\phi(n, g(n))$ holds for all n. For concreteness, we may assume g is the Ackerman function.

Fix a nonempty Π^0_1 class Q_a, with corresponding tree T_a, such that WKL_0 proves Q_a is nonempty. For each $\sigma \in T_a$, we define a sequence

$$\sigma^* = 0^{g(0)}\, 1\, \sigma_0\, 0^{g(1)}\, 1\, \sigma_1\, \cdots\, 0^{g(k)}\, 1\, \sigma_k$$

where $|\sigma| = k + 1$ and 0^r denotes a sequence of r zeros (we let $\langle\rangle^* = \langle\rangle$). Define $T^* = \{\tau \mid \exists \sigma \in T(\tau \subseteq \sigma^*)\}$. It can be seen that T^* is a computable subtree of $2^{<\mathbb{N}}$ and $[T^*] \equiv_M [T]$. Let $b \in \omega$ be an index such that such that $T^* = T_b$. It is important that RCA_0 proves that for each $\tau \in T_b$ there is a $\sigma \in T$ with $\tau \subseteq \sigma^*$; this will be provable if the index b is chosen correctly.

To obtain a contradiction, assume WKL_0 can prove $Q_b \leq_w Q_a$. Thus, since WKL_0 proves Q_a is nonempty, WKL_0 proves Q_b is nonempty. Using Σ^0_1 induction relative to an element of Q_b, WKL_0 proves the Π^0_2 sentence that for all n there is an m such that $g(n) = m$.

The conclusion we reach in the previous paragraph is impossible, because WKL_0 is conservative over PRA for Π^0_2 sentences and g is not primitive recursive. □

3 Applications of a result of Binns and Simpson

In this section, we demonstrate an embedding of the free distributive lattice on ω generators into $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$. We require the following theorem, which is implicit in a paper of Binns and Simpson [2]. We sketch a proof assuming the reader has access to the original paper. The notation $\bigoplus f_i$ denotes the Turing join of a sequence of functions $\langle f_i \mid i \in \omega \rangle$.

Theorem 3.1. Given any nonempty Π^0_1 class $P \subseteq 2^\omega$ with no computable elements, there is an infinite computable sequence $\langle Q_i \subseteq 2^\omega \mid i \in \omega \rangle$ of Π^0_1 classes with the following properties.

1. For any sequence $\langle f_i \mid i \in \omega \rangle$ such that $f_i \in Q_i$ and any $g \in P$, g is not Turing reducible to $\bigoplus f_i$.

2. For any sequence $\langle f_i \mid i \in \omega, i \neq j \rangle$ such that $f_i \in Q_i$ and any $f \in Q_j$, f is not Turing reducible to $\bigoplus f_i$.

Sketch of proof. Begin by letting Q be the Π^0_1 class constructed in Theorem 2.1 of [2]. Split Q into a sequence $\langle Q_i \mid i \in \mathbb{N} \rangle$ as in Theorem 3.1 of that paper. The sequence $\langle Q_i \rangle$ has the desired properties. □

Theorem 3.2. The free distributive lattice with ω generators embeds into $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$.

Proof. Let $\langle Q_i \rangle$ be the sequence of Π^0_1 classes constructed in Theorem 3.1 and let $\langle e(i) \mid i \in \mathbb{N} \rangle$ be a computable sequence of indices such that $Q_i = [T_{e(i)}]$ for each $i \in \mathbb{N}$. For each $i \in \omega$ let ϕ_i be the sentence "Either $T_{e(i)}$ is finite or $[T_{e(i)}]$ is nonempty."

We claim that the sentences $\langle \phi_i \mid i \in \omega \rangle$ generate a free distributive lattice. Given two finite subsets X, Y of ω, we show $(*)$: if $\inf_{i \in X} \phi_i \leq \sup_{j \in Y} \phi_j$ then $X \cap Y \neq \varnothing$. By Theorem II.2.3 of [3], this suffices to show that the lattice generated by $\langle \phi_i \rangle$ is free. To this end, let $X, Y \subseteq \omega$ be finite, let Φ be the conjunction of $\{\phi_i \mid i \in X\}$, and let Ψ be the disjunction of $\{\phi_j \mid j \in Y\}$. Proposition $(*)$ says that if $\Phi \vdash \Psi$ then $X \cap Y$ is nonempty.

Suppose $X \cap Y$ is empty. Let $\langle f_i \mid i \in X \rangle$ be such that $f_i \in Q_i$ for each $i \in X$. Let $M = \{Z \subseteq \omega \mid Z \leq_T \bigoplus_{i \in X} f_i\}$. Then M is an ω-model of $\mathsf{RCA}_0 + \Phi$, and by property (2) of Theorem 3.1, $M \not\models \phi_j$ for any $j \in Y$. Thus if $X \cap Y$ is empty then $\Phi \nvdash \Psi$. □

We draw two corollaries from properties of the free distributive lattice on ω generators which, by the previous theorem, are shared by the lattice $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$. These corollaries also follow from the characterization of $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ as a countable atomless Boolean algebra; Theorem 3.2 shows that the embeddings can be constructed in a particular manner.

Corollary 3.3. Every finite distributive lattice can be embedded into the lattice $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$.

Corollary 3.4. There is a sequence in $\mathcal{A}(\mathsf{WKL}_0, \mathsf{RCA}_0)$ with the order type of the integers.

Remark 3.5. The subsystems constructed in the previous theorem and corollaries are mathematical in the sense that they are stated in terms of the existence of paths through certain trees; there is no metamathematical content to the subsystems. The indices for these trees are defined to implement parts of a priority argument construction, but this construction makes no reference to consistency statements.

In Theorem 10.7 of [7], Simpson constructs a Σ^1_1 formula ϕ that is provable from WKL_0 but not RCA_0. The construction of this formula relies on a sentence expressing the consistency of Σ^0_1 induction. The sentence ϕ is not equivalent to the consistency of Σ^0_1 induction over RCA_0, however, because of conservativity.

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